

On the Design of Attitude Observers on the Special Orthogonal Group $SO(3)$

Soulaimane Berkane and Abdelhamid Tayebi

Abstract—We revisit the nonlinear complimentary filter on $SO(3)$, previously proposed in the literature, and provide the (time-explicit) solution to the matrix ODE governing the attitude estimation error in the absence of measurement errors. The stability and performance properties of this filter can be easily deduced from the obtained closed-form solution. Thereafter, we consider two nonlinear complimentary filters (with state-dependant gains) which are shown to exhibit improved stability and performance proprieties compared to the traditional filter. We perform robustness analysis for the three discussed attitude filters on $SO(3)$ with respect to attitude and angular velocity measurement errors. Specifically, we show that the state-dependant-gain filters may exhibit improved robustness to gyro measurement disturbances and a better disturbance attenuation levels. Simulation results are performed to confirm the obtained theoretical results.

I. INTRODUCTION

The ability to estimate the orientation (attitude) of a rigid body is an important feature in many engineering applications. As such, this problem has attracted the attention of many researchers and industrials for several decades. This is mainly due to the fact that there is “no sensor” that directly measures the attitude. The attitude information is usually reconstructed using a set of body-frame measurements of known inertial vectors. Static attitude reconstruction from inertial vector measurements is one of the earliest solutions to this problem (see, for instance, [1], [2]). Although simple, these methods do not perform well in the presence of measurements noise. As an alternative solution, several Kalman-type filters have been developed and successfully used in aerospace applications, although with extra care as they usually rely on linearizations and heavy computations (see, for instance, [3]–[6]). On the other hand, simple and yet practical linear complementary filters (for small rotational motions) have been successfully used in practical applications, *e.g.*, [7], [8], where the angular velocity is used to complement the inertial vector measurements to improve the estimation accuracy through an appropriate filtering. Nonlinear attitude filters that use the quaternion measurements have been proposed in [9]–[13]. More recently, nonlinear complimentary filters, evolving on $SO(3)$, have emerged and showed their ability in handling efficiently the attitude estimation problem [14]–[20]. These

filters have the distinctive advantage of using directly inertial vector measurements which are available on-board of most aerial and underwater vehicles; thus obviating the need of reconstructing the attitude. This class of smooth nonlinear observers guarantees, in general, *almost* global asymptotic stability (AGAS), *i.e.*, convergence to the actual attitude is guaranteed from any initial condition except from a set of Lebesgue measure zero. As a matter of fact, AGAS is the strongest result one aims to obtain on a compact manifold such as $SO(3)$ using time-invariant continuous control or estimation algorithms [21], [22]. To overcome this topological obstruction, attitude estimators (evolving outside $SO(3)$) with global asymptotic and exponential stability properties have been proposed in [23] and [24], respectively. The topological obstruction on $SO(3)$ has been also successfully addressed via the *synergistic* hybrid technique [25]–[28]. Using this approach, global asymptotic hybrid attitude observers on $SO(3)$ have been proposed in [29] and global exponential hybrid attitude observers on $SO(3)$ have been proposed in [30]–[32].

Recent studies, such as [19], [20], [29], [33], [34], pointed out that the nonlinear complimentary filters proposed in [14], [18], which are widely used in practice, may suffer from slow convergence and robustness issues. Motivated by these recent studies, the present paper aims to conduct a rigorous performance and robustness analysis of the nonlinear complementary filter on $SO(3)$ and proposes different directions and solutions for improvement. First, we revisit the nonlinear complementary filter on $SO(3)$ proposed in [14] in the case of unbiased angular velocity measurements. We derive a closed-form (time-explicit) solution for the estimation error dynamics. The stability and performance properties of this filter can be directly deduced from the obtained solution. In particular, we derive a lower bound on the convergence time of the filter and consequently explain (rigorously) why the filter suffers from slow convergence when initialized at large attitude estimation errors. Then, we consider two state-dependent-gain nonlinear attitude estimators, evolving both on $SO(3)$, exhibiting faster convergence rates, compared to the attitude observer of [14]. The two attitude estimators share a similar structure to the observer proposed in [14] and are very similar, up to some minor details, to the filters proposed in [20] and [34]; which are also inspired from [11], [35]. The two proposed filters are, however, singularity-free compared to [20] and [34]. Note that for the sake of simplicity we ignore the integral bias adaptation law proposed in [14] which can be added in real applications without affecting the stability of the filter as shown in [14]. Furthermore, we investigate the robustness properties of these proposed nonlinear complementary filters

This work was supported by the National Sciences and Engineering Research Council of Canada (NSERC).

The authors are with the Department of Electrical and Computer Engineering, University of Western Ontario, London, Ontario, Canada. A. Tayebi is also with the Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, Canada. sberkane@uwo.ca, atayebi@lakeheadu.ca

on $SO(3)$ in the presence of bounded gyro measurement errors and small attitude measurement errors. It is shown that the newly proposed attitude filters exhibit larger robustness domains compared to the traditional constant gain filter.

II. BACKGROUND AND PRELIMINARIES

Throughout the paper, we use \mathbb{R} and \mathbb{R}^+ to denote, respectively, the sets of real and nonnegative real numbers. The Euclidean norm of $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^\top x}$. For a square matrix $A \in \mathbb{R}^{n \times n}$, we denote by λ_i^A , λ_{\min}^A , and λ_{\max}^A the i th, minimum, and maximum eigenvalue of A , respectively. The rigid body attitude evolves on the Special Orthogonal group defined as $SO(3) := \{R \in \mathbb{R}^{3 \times 3} \mid \det(R) = 1, RR^\top = I\}$, where I is the three-dimensional identity matrix and $R \in SO(3)$ is called a *rotation matrix*. The *Lie algebra* of $SO(3)$, denoted by $\mathfrak{so}(3) := \{\Omega \in \mathbb{R}^{3 \times 3} \mid \Omega^\top = -\Omega\}$, is the vector space of 3-by-3 skew-symmetric matrices. Let the map $[\cdot]_\times : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ be defined such that $[x]_\times y = x \times y$, for any $x, y \in \mathbb{R}^3$, where \times is the vector cross-product on \mathbb{R}^3 . Let $\text{vex} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ denote the inverse isomorphism of the map $[\cdot]_\times$, such that $\text{vex}([\omega]_\times) = \omega$, for all $\omega \in \mathbb{R}^3$ and $[\text{vex}(\Omega)]_\times = \Omega$, for all $\Omega \in \mathfrak{so}(3)$. Defining $\mathbb{P}_a : \mathbb{R}^{3 \times 3} \rightarrow \mathfrak{so}(3)$ as the projection map on the Lie algebra $\mathfrak{so}(3)$ such that $\mathbb{P}_a(A) := (A - A^\top)/2$, we can extend the definition of vex to $\mathbb{R}^{3 \times 3}$ by taking the composition map $\psi := \text{vex} \circ \mathbb{P}_a$ such that, for a 3-by-3 matrix $A := [a_{ij}]_{i,j=1,2,3}$, one has

$$\psi(A) := \text{vex}(\mathbb{P}_a(A)) = \frac{1}{2} \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}. \quad (1)$$

Let $|R|_I \in [0, 1]$ be the normalized Euclidean distance on $SO(3)$ which is given by

$$|R|_I^2 := \frac{1}{4} \text{tr}(I - R). \quad (2)$$

The attitude of a rigid body can also be represented as a rotation of angle $\theta \in \mathbb{R}$ around a unit vector axis $u \in \mathbb{S}^2$. This is commonly known as the angle-axis parametrization of $SO(3)$ and is given by the map $\mathcal{R}_a : \mathbb{R} \times \mathbb{S}^2 \rightarrow SO(3)$ such that

$$\mathcal{R}_a(\theta, u) = I + \sin(\theta)[u]_\times + (1 - \cos(\theta))[u]_\times^2. \quad (3)$$

Alternatively, elements of $SO(3)$ can be parameterized by vectors on \mathbb{R}^3 through the map $\mathcal{R}_r : \mathbb{R}^3 \rightarrow SO(3)$ such that

$$\begin{aligned} \mathcal{R}_r(z) &= (I + [z]_\times)(I - [z]_\times)^{-1} \\ &= \frac{1}{1 + \|z\|^2} ((1 - \|z\|^2)I + 2zz^\top + 2[z]_\times). \end{aligned} \quad (4)$$

Equation (4) is often known as Cayley's formula [36]. Note that the matrix $(I - [z]_\times)$ is always invertible for all $z \in \mathbb{R}^3$. In fact, since $[z]_\times$ is a skew-symmetric matrix, all its eigenvalues are pure imaginary and, thus, all the eigenvalues of $I - [z]_\times$ are non-zero. The map \mathcal{R}_r is a diffeomorphism between \mathbb{R}^3 and $SO(3) \setminus \Pi$ with $\Pi = \{R \in SO(3) \mid |R|_I = 1\}$. The inverse map $\mathcal{Z} : SO(3) \setminus \Pi \rightarrow \mathbb{R}^3$ is given by

$$\mathcal{Z}(R) = \text{vex}((R - I)(R + I)^{-1}) = \frac{\psi(R)}{2(1 - |R|_I^2)}. \quad (5)$$

The vector $\mathcal{Z}(R) \in \mathbb{R}^3$ defines the vector of Rodrigues parameters. Note that the Rodrigues vector is usually defined using unit quaternion or the angle-axis representation[37]. We prefer to use directly rotation matrices on $SO(3)$ in (5). It can be verified that the time derivative of the Rodrigues vector $\mathcal{Z}(R)$ along the trajectories of $\dot{R} = R[\xi]_\times$, $\xi \in \mathbb{R}^3$ is given by

$$\frac{d}{dt}\mathcal{Z}(R) = \frac{1}{2}(I + [\mathcal{Z}(R)]_\times + \mathcal{Z}(R)\mathcal{Z}(R)^\top)\xi. \quad (6)$$

It is worth pointing out that all the attitude filters derived in this paper are directly evolving on the Special Orthogonal group $SO(3)$. The introduction of the three-parameters Rodrigues vector is only for the sake of analysis. The following technical lemmas are useful throughout the paper.

Lemma 1. *Let $R \in SO(3)$ and $A = A^\top \in \mathbb{R}^{3 \times 3}$ such that $\bar{A} = \frac{1}{2}(\text{tr}(A)I - A)$ is positive definite. Then, the following hold*

$$\|\psi(R)\|^2 = |R|_I^2 = 4|R|_I^2(1 - |R|_I^2), \quad (7)$$

$$4\lambda_{\min}^{\bar{A}}|R|_I^2 \leq \text{tr}(A(I - R)) \leq 4\lambda_{\max}^{\bar{A}}|R|_I^2, \quad (8)$$

$$\xi^2|R|_I^2(1 - |R|_I^2) \leq \frac{\|\psi(AR)\|^2}{(2\lambda_{\max}^{\bar{A}})^2} \leq |R|_I^2(1 - \xi^2|R|_I^2) \quad (9)$$

such that $\xi := \lambda_{\min}^{\bar{A}}/\lambda_{\max}^{\bar{A}}$. Moreover, for all $R \in SO(3) \setminus \Pi$,

$$\psi(AR) = \frac{2(I - [\mathcal{Z}(R)]_\times)}{1 + \|\mathcal{Z}(R)\|^2} \bar{A}\mathcal{Z}(R). \quad (10)$$

Proof. See Appendix A □

The following definition and characterization of Local-Input-to-State-Stability (LISS) property for nonlinear systems is needed throughout the paper and can be found in [38]. Consider the system

$$\dot{x} = f(x, u), \quad (11)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz in x and u . The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$.

Definition 1. *System 11 is said to be locally input-to-state stable if there exist $k_x, k_u > 0$, $\gamma \in \mathcal{K}$, $\beta \in \mathcal{KL}$ such that*

$$\begin{aligned} \|x(0)\| < k_x \text{ and } \sup_{t \geq 0} \|u(t)\| < k_u \Rightarrow \\ \|x(t)\| &\leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{t \geq 0} \|u(t)\|\right), \forall t \geq 0. \end{aligned} \quad (12)$$

Lemma 2. *Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad (13)$$

$$\dot{V}(x) \leq -\alpha_3(\|x\|), \forall \|x\| \geq \rho(\|u\|), \quad (14)$$

for all $\|x\| < r_x$ and $\|u\| < r_u$, where $\alpha_i, i = 1, 2, 3$ and ρ are class \mathcal{K} functions. Then, the system (11) is locally input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$, $k_x = \alpha_2^{-1} \circ \alpha_1(r_x)$ and $k_u = \min\{\rho^{-1}(k_x), r_u\}$.

III. THE ATTITUDE ESTIMATION PROBLEM

Let $R \in SO(3)$ denote a rotation matrix from the body fixed-frame to a given inertial reference frame. The rotation matrix R evolves according to the kinematic equation

$$\dot{R} = R[\omega]_{\times}, \quad (15)$$

where $\omega \in \mathbb{R}^3$ is the angular velocity expressed in the body fixed-frame. Let $\omega_y(t)$ denote the angular velocity measurement (usually provided by a gyroscope) such that

$$\omega_y(t) = \omega(t) + n_\omega(t), \quad (16)$$

where $n_\omega(t)$ is *a priori* bounded signal that captures the gyro-bias, measurements noise and all other disturbances. Attitude information is usually extracted from body-frame measurements of known reference vectors such as those obtained from accelerometers, magnetometers or star trackers. Two non-collinear vector measurements are usually sufficient to provide an algebraic reconstruction of the attitude matrix, namely $R_y(t)$ such that

$$R_y(t) = N_R(t)R(t), \quad (17)$$

where $N_R(t) \in SO(3)$ is a rotation matrix that captures all the perturbations and measurement errors that are inherent to the attitude reconstruction procedure at hand. Many (static) attitude reconstruction schemes are available, see for instance the TRIAD [39], the SVD [2] and the QUEST [1]. The reconstructed attitude $R_y(t)$ is not reliable in practical applications due to measurements noise and the limited bandwidth (and sometimes the poor quality) of the inertial sensors [40].

The goal of the attitude complementary filter, is to fuse the available gyro measurements together with the reconstructed attitude R_y (or directly the inertial vector measurements) to obtain a good (filtered) attitude estimate \hat{R} . Note that, throughout this paper, we will use the terms ‘filter’, ‘estimator’ and ‘observer’ indistinguishably.

IV. PERFORMANCE ANALYSIS FOR DIFFERENT NONLINEAR COMPLIMENTARY FILTERS ON $SO(3)$ IN THE ABSENCE OF MEASUREMENT ERRORS

In this section, we study the stability and performance of three different deterministic nonlinear complementary attitude filters on $SO(3)$ in the case of perfect measurements, *i.e.* we consider

$$R_y(t) = R(t) \text{ and } \omega_y(t) = \omega(t), \quad \forall t \geq 0. \quad (18)$$

Although, the introduction of a filter is not necessary under the above assumption, the study of the dynamics of a given filtering scheme is conducted in the error-free case to avoid the complexities introduced by considering random noise and disturbances in the sensors measurements. In the next section, however, we will study the robustness property of the proposed filtering methods in the presence of measurement errors.

The first discussed filter is inspired from [14], [40] where we, for the sake of simplicity, ignore the angular velocity bias vector. The two other filters are similar in their structure to the filter in [14] up to a state-dependent-gain in the filter innovation term. To this purpose, we derive closed-form

solutions to the differential equations governing the attitude estimation error. We believe that these results as well as the analysis methods used to conclude the stability and the explicit performance of the filters are novel.

Consider the following nonlinear complementary attitude filter on $SO(3)$ inspired from Mahony *et al.* [14]:

$$\text{Filter I} \quad \begin{cases} \dot{\hat{R}} &= \hat{R}[\omega_y]_{\times} - [\sigma]_{\times} \hat{R}, \\ \sigma &= -\psi(A\hat{R}_y\hat{R}^\top), \end{cases} \quad (19)$$

where $\hat{R} \in SO(3)$ is an estimate of R with $\hat{R}(0) = \hat{R}_0 \in SO(3)$ and A is a symmetric matrix such that $\tilde{A} := \frac{1}{2}(\text{tr}(A)I - A)$ is positive definite. The attitude estimator (19) falls under the category of gradient-based observers on Lie groups [41]. In fact, for $R_y \equiv R$, the observer innovation term $\sigma = -\psi(A\hat{R})$, where $\tilde{R} = R\hat{R}^\top$ represents the attitude estimation error, can be directly obtained from the gradient of the following smooth attitude potential function on $SO(3)$

$$U_A(\tilde{R}) = \text{tr}(A(I - \tilde{R})), \quad (20)$$

which is the well-known weighted trace function on $SO(3)$ that has been widely used in the literature for the design of attitude control systems [21], [28], [42]. The attitude filter (19) has been used in many academic and industrial applications due to its proven almost global asymptotic stability, local exponential stability and its nice filtering properties. In the following theorem we give (for the first time) the explicit solution of the attitude estimator (19).

Theorem 1. *Consider the attitude kinematics system (15) coupled with the attitude observer (19) under assumption (18). Then,*

- *The closed-loop system has at least four equilibria characterized by $\{I\} \cup \mathcal{R}_a(\pi, \mathcal{E}(A))$ where $\mathcal{E}(A)$ is the set of all unit eigenvectors of A .*
- *The set of all rotations of angle π , defined by $\Pi = \{\tilde{R} \in SO(3) \mid |\tilde{R}|_I = 1\}$, is invariant and non-attractive.*
- *For any $\tilde{R}(0) \in SO(3) \setminus \Pi$, one has*

$$\tilde{R}(t) = \mathcal{R}_r(e^{-\tilde{A}t} \mathcal{Z}(\tilde{R}(0))), \quad \forall t \geq 0. \quad (21)$$

Proof. The proof of the first two items of Theorem 1 can be directly deduced from [43, Proposition 1]. Nevertheless, for the sake of completeness, we prove these properties again. Using the product rule, and in view of (15) and (19), one obtains

$$\begin{aligned} \dot{\tilde{R}} &= \dot{R}\hat{R}^\top - R\hat{R}^\top\dot{\hat{R}}\hat{R}^\top \\ &= R[\omega]_{\times}\hat{R}^\top - R[\omega]_{\times}\hat{R}^\top + R\hat{R}^\top[\sigma]_{\times} \\ &= \tilde{R}[\sigma]_{\times}, \end{aligned} \quad (22)$$

where the fact that $[u]_{\times}P^\top = P^\top[Pu]_{\times}$, for all $u \in \mathbb{R}^3$ and $P \in SO(3)$, has been used to obtain the last equality. The equilibria of the closed loop-system are characterized by $\sigma = -\psi(A\tilde{R}) = -\text{vex}(\mathbb{P}_a(A\tilde{R})) = 0$ which by [28, Lemma 2] implies that $\tilde{R} \in \{I\} \cup \mathcal{R}_a(\pi, \mathcal{E}(A))$. Now, since $|\tilde{R}|_I^2 =$

$\text{tr}(I - \tilde{R})/4$, it follows that the time derivative of $|\tilde{R}|_I^2$ along the trajectories of (22) satisfies

$$\begin{aligned} \frac{d}{dt}|\tilde{R}|_I^2 &= -\text{tr}(\tilde{R}[\sigma]_{\times})/4 \\ &= \text{tr}(\tilde{R}\mathbb{P}_a(A\tilde{R}))/4 \\ &= \text{tr}(\tilde{R}(A\tilde{R} - \tilde{R}^{\top}A))/8 \\ &= -\text{tr}(A(I - \tilde{R}^2))/8. \end{aligned} \quad (23)$$

Therefore, in view of (7)-(8), it follows that

$$-2\lambda_{\max}^{\bar{A}}(1 - |\tilde{R}|_I^2)|\tilde{R}|_I^2 \leq \frac{d}{dt}|\tilde{R}|_I^2 \leq -2\lambda_{\min}^{\bar{A}}(1 - |\tilde{R}|_I^2)|\tilde{R}|_I^2, \quad (24)$$

which shows that the set Π is forward invariant and a repeller.

Now, assume that $\tilde{R}(0) \in SO(3) \setminus \Pi$ which implies, in view of the fact that Π is a repeller, that $\tilde{R}(t) \in SO(3) \setminus \Pi$ for all future time $t \geq 0$. Therefore, the inverse map $\mathcal{Z}(\tilde{R})$, defined in (5), exists for all $t \geq 0$ such that one has $\mathcal{R}_r(\mathcal{Z}(\tilde{R}(t))) = \tilde{R}(t)$. Making use of (6), (10) and (22) one obtains

$$\begin{aligned} \frac{d}{dt}\mathcal{Z}(\tilde{R}) &= \frac{1}{2}(I + [\mathcal{Z}(\tilde{R})]_{\times} + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^{\top})\sigma \\ &= -\frac{1}{2}(I + [\mathcal{Z}(\tilde{R})]_{\times} + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^{\top})\psi(A\tilde{R}) \\ &= -(I + [\mathcal{Z}(\tilde{R})]_{\times} + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^{\top})\frac{(I - [\mathcal{Z}(\tilde{R})]_{\times})}{1 + \|\mathcal{Z}(\tilde{R})\|^2}\bar{A}\mathcal{Z}(\tilde{R}) \\ &= -\frac{1}{(1 + \|\mathcal{Z}(\tilde{R})\|^2)}(I - [\mathcal{Z}(\tilde{R})]_{\times}^2 + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^{\top})\bar{A}\mathcal{Z}(\tilde{R}) \\ &= -\bar{A}\mathcal{Z}(\tilde{R}), \end{aligned} \quad (25)$$

where we have used the fact that $[\mathcal{Z}(\tilde{R})]_{\times}^2 = -\|\mathcal{Z}(\tilde{R})\|^2 I + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^{\top}$ to obtain the last equality. By simple integration of (25), it follows that

$$\mathcal{Z}(\tilde{R}(t)) = e^{-\bar{A}t}\mathcal{Z}(\tilde{R}(0)), \quad (26)$$

for all $t \geq 0$, which yields (21). \square

Theorem 1 provides an explicit solution for the attitude estimator of Mahony *et al.* [14], given by equations (19), in the absence of measurement errors. Equation (25) shows that the Rodrigues vector associated to the attitude estimation error follows the dynamics of a linear time-invariant system with negative definite state matrix. The three-parameters Rodriguez vector decays, therefore, exponentially fast and the explicit solution for the linear system can be derived as in (26). The corresponding attitude error matrix is subsequently obtained via the Cayley's formula (21). It is worth pointing out that, although the Rodrigues vector is converging exponentially to zero, the attitude estimation error does not necessary converge exponentially fast as well. The convergence property of the norm of the attitude error is given in the following corollary.

Corollary 1. *Consider the attitude kinematics system (15) coupled with the attitude observer (19) under assumption (18).*

Then, for any $\tilde{R}(0) \in SO(3) \setminus \Pi$, the Euclidean distance of the attitude error \tilde{R} on $SO(3)$ is given by

$$|\tilde{R}(t)|_I^2 = \frac{\psi(\tilde{R}(0))^{\top} e^{-2\bar{A}t} \psi(\tilde{R}(0))}{4(1 - |\tilde{R}(0)|_I^2) + \psi(\tilde{R}(0))^{\top} e^{-2\bar{A}t} \psi(\tilde{R}(0))}, \quad (27)$$

for all $t \geq 0$.

Proof. In view of (5), (7) it follows that

$$\|\mathcal{Z}(\tilde{R}(t))\|^2 = \frac{\|\psi(\tilde{R}(t))\|^2}{4(1 - |\tilde{R}(t)|_I^2)} = \frac{|\tilde{R}(t)|_I^2}{1 - |\tilde{R}(t)|_I^2}. \quad (28)$$

On the other hand, using the result of Theorem 1, one has

$$\begin{aligned} \|\mathcal{Z}(\tilde{R}(t))\|^2 &= \|e^{-\bar{A}t}\mathcal{Z}(\tilde{R}(0))\|^2 \\ &= \frac{\psi(\tilde{R}(0))^{\top} e^{-2\bar{A}t} \psi(\tilde{R}(0))}{4(1 - |\tilde{R}(0)|_I^2)^2}, \end{aligned} \quad (29)$$

where (5) has been used to obtain the last equality. Simple algebraic manipulation of (28) and (29) yields (27). \square

Corollary 1 provides an explicit expression showing the evolution of the Euclidean distance $|\tilde{R}(t)|_I$ with respect to time. Note that it is not difficult to show that the vector $\psi(\tilde{R}) = \sin(\theta)u$ where $\tilde{R} = \mathcal{R}_a(\theta, u)$. Therefore, from (27), for the same initial attitude error angle, initial attitude errors with rotation axis $u(0)$ in the direction of the larger spectrum of \bar{A} tends to generate larger attitude errors $|\tilde{R}(t)|_I$ compared to an initial attitude error with rotation axis $u(0)$ in the direction of a smaller spectrum (eigenvalue) of \bar{A} . The study of the effect of the initial attitude angle on the performance of the attitude filter (19) is provided in the result of the following Corollary.

Corollary 2. *Consider the attitude kinematics system (15) coupled with the attitude observer (19) under assumption (18). Then, for any $\tilde{R}(0) \in SO(3) \setminus \Pi$, the attitude estimation error satisfies*

$$\underline{\beta}(|\tilde{R}(0)|_I, t) \leq |\tilde{R}(t)|_I \leq \bar{\beta}(|\tilde{R}(0)|_I, t), \quad (30)$$

for all $t \geq 0$, such that $\underline{\beta}$ and $\bar{\beta}$ are given by

$$\begin{aligned} \bar{\beta}(s, t) &= \frac{se^{-\lambda_{\min}^{\bar{A}}t}}{(1 - s^2(1 - e^{-2\lambda_{\min}^{\bar{A}}t}))^{\frac{1}{2}}}, \\ \underline{\beta}(s, t) &= \frac{se^{-\lambda_{\max}^{\bar{A}}t}}{(1 - s^2(1 - e^{-2\lambda_{\max}^{\bar{A}}t}))^{\frac{1}{2}}}. \end{aligned}$$

Proof. First, it should be noted that the matrix $e^{-\bar{A}t}$ is positive definite due to the fact that the matrix $-\bar{A}$ is symmetric. Moreover, the eigenvalues of the matrix $e^{-\bar{A}t}$ are given by $e^{-\lambda_i^{\bar{A}}t}$, $i = 1, 2, 3$, where $\lambda_i^{\bar{A}}$, $i = 1, 2, 3$, are the eigenvalues of \bar{A} . Hence, in view of (7), one has

$$\begin{aligned} \psi(\tilde{R}(0))^{\top} e^{-2\bar{A}t} \psi(\tilde{R}(0)) &\leq e^{-2\lambda_{\min}^{\bar{A}}t} \|\psi(\tilde{R}(0))\|^2 \\ &\leq 4e^{-2\lambda_{\min}^{\bar{A}}t} |\tilde{R}(0)|_I^2 (1 - |\tilde{R}(0)|_I^2), \end{aligned}$$

which, in view of (27) and the fact that the map $x \rightarrow x/(x+a)$ is non-decreasing for all $a \geq 0$, implies that

$$|\tilde{R}(t)|_I^2 \leq \frac{e^{-2\lambda_{\min}^{\bar{A}}t} |\tilde{R}(0)|_I^2}{1 - |\tilde{R}(0)|_I^2 + e^{-2\lambda_{\min}^{\bar{A}}t} |\tilde{R}(0)|_I^2} = (\bar{\beta}(|\tilde{R}(0)|_I, t))^2.$$

Following similar steps as above, the following lower bound can be derived

$$|\tilde{R}(t)|_I^2 \geq \frac{e^{-2\lambda_{\max}^A t} |\tilde{R}(0)|_I^2}{1 - |\tilde{R}(0)|_I^2 + e^{-2\lambda_{\max}^A t} |\tilde{R}(0)|_I^2} = (\beta(|\tilde{R}(0)|_I, t))^2.$$

□

According to the upper bound on the estimation error given in Corollary 2, it is clear that for small initial conditions, i.e., $|\tilde{R}(0)|_I \ll 1$, the attitude estimation error satisfies $|\tilde{R}(t)|_I \leq |\tilde{R}(0)|_I \exp(-\lambda_{\min}^A t)$ which confirms the local exponential stability of the equilibrium point $|\tilde{R}|_I = 0$ proved in [14]. Moreover, the convergence rate of the filter is given in the following corollary

Corollary 3. *Starting from any initial condition $\tilde{R}(0) \in SO(3) \setminus \Pi$, the time t_B necessary to enter the ball of radius $|\tilde{R}(t)|_I = B$ satisfies*

$$t_B \geq \frac{1}{\lambda_{\max}^A} \ln \left(\frac{|\tilde{R}(0)|_I (1 - B^2)^{\frac{1}{2}}}{B(1 - |\tilde{R}(0)|_I^2)^{\frac{1}{2}}} \right). \quad (31)$$

Proof. Using the lower bound of (30), the time t_B needs to satisfy the constraint

$$\beta(|\tilde{R}(0)|_I, t_B) \leq |\tilde{R}(t)|_I = B.$$

Using straightforward algebraic manipulations, the above inequality reads

$$e^{-\lambda_{\max}^A t_B} \leq \frac{B^2(1 - |\tilde{R}(0)|_I^2)}{|\tilde{R}(0)|_I^2(1 - B^2)},$$

which leads to the result of the corollary by taking the $\ln(\cdot)$ function on both sides of the above inequality. □

Therefore, according to Corollary 3, it is clear that large initial estimation errors, i.e., $|\tilde{R}(0)|_I \rightarrow 1$, will result in low convergence rates. This fact has been numerically and experimentally observed in recent works such as [20], [29], [35]

In a tentative to improve the convergence rate of this class of attitude observers, we introduce a state-dependent scalar gain function $k : SO(3) \rightarrow \mathbb{R}^+ \setminus \{0\}$, into the observer innovation term such that $\sigma_k = -k(\tilde{R})\psi(A\tilde{R})$ under the following assumptions:

- The scalar function $k(\cdot)$ is strictly positive on $SO(3)$.
- The scalar function $k(\cdot)$ is a priori *bounded* function on $SO(3)$.
- The scalar function $k(\cdot)$ is large enough for large attitude errors \tilde{R} (for errors such that $|\tilde{R}|_I \rightarrow 1$).

The assumption that $k(\tilde{R})$ is strictly positive is necessary to preserve the stability and convergence of the nonlinear complementary filter. In fact, following similar steps as in (23)-(24) one can show that, with the introduction of the new innovation term σ_k , one has $d|\tilde{R}|_I^2/dt \leq -2\lambda_{\min}^A k(\tilde{R})(1 - |\tilde{R}|_I^2)|\tilde{R}|_I^2 \leq -2\lambda_{\min}^A \underline{k}(1 - |\tilde{R}|_I^2)|\tilde{R}|_I^2$ where $\underline{k} > 0$ is a lower bound on $k(\tilde{R})$. The assumption that $k(\tilde{R})$ is bounded for all $\tilde{R} \in SO(3)$ is needed for practical implementation while the assumption that $k(\tilde{R})$ is large for large attitude errors is introduced to increase the convergence rate for large errors.

Note that the innovation term σ of Filter I is equals σ_{k_1} with a gain function $k_1(\tilde{R}) = 1$ for all $\tilde{R} \in SO(3)$.

The nonlinear complementary filter which has been proposed recently in [20] can be obtain by taking the innovation term σ_k with the gain function $k(\tilde{R}) = [4\lambda_{\min}^A - U_A(\tilde{R})]^{-\frac{1}{2}}$ where U_A is given by (20) for some matrix A . Although, the resulting filter in [20] guarantees faster convergence rates compared to [14] and is written directly in terms of vector measurements (no need for rotation matrix reconstruction), the implementation of [20] is constrained on the *ellipsoid-like* set

$$\mathcal{S}_{\min} = \{\tilde{R} \in SO(3) \mid U_A(\tilde{R}) < 4\lambda_{\min}^A\}. \quad (32)$$

In view of (8), the largest ball contained in the above defined ellipsoid is given by

$$\mathcal{B}_\xi = \{\tilde{R} \in SO(3) \mid |\tilde{R}|_I^2 < \xi\}, \quad (33)$$

where $\xi = \lambda_{\min}^A / \lambda_{\max}^A$. The ball \mathcal{B}_ξ shrinks as the spread of the eigenvalues gets larger, i.e., as ξ gets smaller. This fact limits the applicability of the attitude estimation scheme [20] when the eigenvalues spread ξ is small; at least without considering farther modifications. Another possible choice is to consider the following gain function instead $k(\tilde{R}) = [4\lambda_{\max}^A - U_A(\tilde{R})]^{-\frac{1}{2}}$, which is well defined on the set

$$\mathcal{S}_{\max} = \{\tilde{R} \in SO(3) \mid U_A(\tilde{R}) < 4\lambda_{\max}^A\}, \quad (34)$$

which, in view of (8), contains the set of all attitude errors except attitude errors of angle π , defined by the ball $\mathcal{B}_1 = SO(3) \setminus \Pi$. Note that when A has distinct eigenvalues, the set \mathcal{S}_{\max} reduces to the whole space $SO(3)$ except only one single rotation given by $\mathcal{R}_a(\pi, v_{\max})$ with v_{\max} being the unit eigenvector corresponding to the eigenvalue λ_{\max}^A . However, small convergence rates are expected for large attitude errors in the direction of λ_{\min}^A when the spread of the eigenvalues is large. In fact, if one has $\tilde{R} = \mathcal{R}_a(\pi, v_{\min})$, with v_{\min} being the unit eigenvector corresponding to the eigenvalue λ_{\min}^A , one has $U_A(\tilde{R}) = 4\lambda_{\min}^A$ and therefore $k(\tilde{R}) = [4\lambda_{\max}^A - 4\lambda_{\min}^A]^{-\frac{1}{2}}$ which might be small if the spread of the eigenvalues is large.

In this work, we consider the following choice for the gain function $k_2(\tilde{R}) = [1 + \epsilon - |\tilde{R}|_I^2]^{-\frac{1}{2}}$ where $\epsilon > 0$ is some small enough parameter. The function $k_2(\tilde{R})$ satisfies all the aforementioned assumptions (positive, bounded and large for large \tilde{R}). This results in the following *state-dependent-gain* nonlinear complementary filter on $SO(3)$

$$\text{Filter II} \quad \begin{cases} \dot{\hat{R}} &= \hat{R}[\omega_y]_\times - [\sigma_{k_2}]_\times \hat{R}, \\ \sigma_{k_2} &= -k_2(R_y \hat{R}^\top) \psi(A R_y \hat{R}^\top), \end{cases} \quad (35)$$

where $\hat{R} \in SO(3)$ is an estimate of R with $\hat{R}(0) = \hat{R}_0 \in SO(3)$ and A is a symmetric matrix such that $\bar{A} := \frac{1}{2}(\text{tr}(A)I - A)$ is positive definite.

Theorem 2. *Consider the attitude kinematics system (15) coupled with the attitude observer (35) under assumption (18). Then, $\forall \epsilon, \gamma > 0$ such that $\gamma < [1 + \epsilon]^{-\frac{1}{2}}$, and for any $\tilde{R}(0) \in \mathcal{B}_{\xi_0}$ such that $\xi_0 = 1 - \gamma^2 \epsilon / (1 - \gamma^2)$, one has*

$$\beta(|\tilde{R}(0)|_I, t) \leq |\tilde{R}(t)|_I \leq \bar{\beta}(|\tilde{R}(0)|_I, t), \quad (36)$$

for all $t \geq 0$, such that $\underline{\beta}$ and $\bar{\beta}$ are class \mathcal{KL} functions and are given by

$$\begin{aligned}\bar{\beta}(s, t) &= \frac{s}{\cosh(\gamma \lambda_{\min}^{\bar{A}} t) + (1 - s^2)^{\frac{1}{2}} \sinh(\gamma \lambda_{\min}^{\bar{A}} t)}, \\ \underline{\beta}(s, t) &= \frac{s}{\cosh(\gamma \lambda_{\max}^{\bar{A}} t) + (1 - s^2)^{\frac{1}{2}} \sinh(\gamma \lambda_{\max}^{\bar{A}} t)}.\end{aligned}$$

Proof. Let $\epsilon, \gamma > 0$ such that $\gamma \leq [1 + \epsilon]^{-\frac{1}{2}}$. Then, one can verify that the scalar $\xi_0 = 1 - \gamma^2 \epsilon / (1 - \gamma^2)$ is non-negative. Therefore, the ball \mathcal{B}_{ξ_0} is well defined and non-empty. Let $\tilde{R}(0) \in \mathcal{B}_{\xi_0}$. Following similar steps as in (23)-(24) and in view of the fact that $\sigma = -\psi(A\tilde{R}) / (1 + \epsilon - |\tilde{R}|_I^2)^{\frac{1}{2}}$, one obtains

$$\begin{aligned}-2\lambda_{\max}^{\bar{A}} \frac{1 - |\tilde{R}|_I^2}{(1 + \epsilon - |\tilde{R}|_I^2)^{\frac{1}{2}}} |\tilde{R}|_I^2 &\leq \frac{d}{dt} |\tilde{R}|_I^2 \\ &\leq -2\lambda_{\min}^{\bar{A}} \frac{1 - |\tilde{R}|_I^2}{(1 + \epsilon - |\tilde{R}|_I^2)^{\frac{1}{2}}} |\tilde{R}|_I^2.\end{aligned}\quad (37)$$

Therefore, the attitude error $|\tilde{R}(t)|_I^2$ is strictly decaying on \mathcal{B}_{ξ_0} which implies that \mathcal{B}_{ξ_0} is forward invariant and, hence, $\tilde{R}(t) \in \mathcal{B}_{\xi_0}$ for all $t \geq 0$. This implies that one has

$$0 \leq |\tilde{R}(t)|_I^2 < 1 - \gamma^2 \epsilon / (1 - \gamma^2) < 1, \quad \forall t \geq 0,$$

which, after few algebraic manipulations, leads to

$$\gamma < \frac{(1 - |\tilde{R}(t)|_I^2)^{\frac{1}{2}}}{(1 + \epsilon - |\tilde{R}(t)|_I^2)^{\frac{1}{2}}} < 1, \quad \forall t \geq 0.$$

It follows from (42) that

$$\begin{aligned}-2\lambda_{\max}^{\bar{A}} (1 - |\tilde{R}|_I^2)^{\frac{1}{2}} |\tilde{R}|_I^2 &\leq \frac{d}{dt} |\tilde{R}|_I^2 \\ &\leq -2\gamma \lambda_{\min}^{\bar{A}} (1 - |\tilde{R}|_I^2)^{\frac{1}{2}} |\tilde{R}|_I^2.\end{aligned}\quad (38)$$

Now making use of the following integral formula

$$\int \frac{dx}{2x(1-x)^{\frac{1}{2}}} = -\operatorname{arctanh}(\sqrt{1-x}) := f(x),$$

and the comparison lemma, one obtains

$$-\lambda_{\max}^{\bar{A}} t \leq f(|\tilde{R}(t)|_I^2) - f(|\tilde{R}(0)|_I^2) \leq -\gamma \lambda_{\min}^{\bar{A}} t. \quad (39)$$

The inverse function f^{-1} is explicitly given by

$$f^{-1}(y) = 1 - \tanh^2(y) = \frac{1}{\cosh^2(y)}.$$

Moreover, using the following identities

$$\begin{aligned}\cosh(a+b) &= \cosh(a) \cosh(b) + \sinh(a) \sinh(b), \\ \cosh(\operatorname{artanh}(x)) &= 1/\sqrt{1-x^2}, \\ \sinh(\operatorname{artanh}(x)) &= x/\sqrt{1-x^2},\end{aligned}$$

it follows that the attitude error $|\tilde{R}(t)|^2$ satisfies

$$\begin{aligned}|\tilde{R}(t)|_I^2 &\leq f^{-1}(-\gamma \lambda_{\min}^{\bar{A}} t + f(|\tilde{R}(0)|_I^2)) \\ &\leq \frac{|\tilde{R}(0)|_I^2}{[\cosh(\gamma \lambda_{\min}^{\bar{A}} t) + (1 - |\tilde{R}(0)|_I^2)^{\frac{1}{2}} \sinh(\gamma \lambda_{\min}^{\bar{A}} t)]^2} \\ &= (\bar{\beta}(|\tilde{R}(0)|_I, t))^2.\end{aligned}$$

The proof is complete. \square

According to Theorem 2, the equilibrium point $|\tilde{R}|_I = 0$ is asymptotically stable inside the ball \mathcal{B}_{ξ_0} . Moreover, using the facts that $\cosh(x) = (e^x - e^{-x})/2$ and $e^{\frac{x}{2}} / (e^x - e^{-x}) \leq 3^{\frac{3}{4}}/4$, it follows that

$$\bar{\beta}(|\tilde{R}(0)|_I, t) \leq \frac{|\tilde{R}(0)|_I}{\cosh(\gamma \lambda_{\min}^{\bar{A}} t)} \leq \frac{3^{\frac{3}{4}}}{4} |\tilde{R}(0)|_I e^{-\gamma \lambda_{\min}^{\bar{A}} t/2}.$$

Hence, the convergence type of Filter II is indeed exponential inside the ball \mathcal{B}_{ξ_0} . Note that when ϵ is chosen sufficiently small such that $\epsilon \rightarrow 0$, one has $\xi_0 \rightarrow 1$ for any $\gamma < [1 + \epsilon]^{-\frac{1}{2}}$. Therefore as the parameter $\epsilon \rightarrow 0$, the region of exponential stability extends to the ball \mathcal{B}_1 which is equivalent to the space of all rotations less than π angle, namely $SO(3) \setminus \Pi$. Note that the scalar γ which appears in the exponential decay factor is *independent* on the initial conditions compared to the smooth attitude estimator (19). This results in faster convergence rates for large attitude errors.

It is worth pointing out that the choice of the innovation term σ_{k_2} in (35) does not correspond, as far as we know, to any gradient of a potential function on $SO(3)$. In fact, this observer was designed by inspection of the dynamics of the attitude error and the desirable performance instead of the traditional systematic gradient-based method where the designer starts from a given potential function, which is typically taken as a Lyapunov candidate, and then designs the observer based on the gradient of this potential function. Our approach proposed above gives a new insight into the design of observers on $SO(3)$, in particular, and for kinematic systems on Lie groups in general.

Nevertheless, if we let $\epsilon \rightarrow 0$ and take $A = I$, it can be shown that σ_{k_2} in (35) is related to the gradient of the non-differentiable potential function $\Phi(\tilde{R}) = 1 - [1 - |\tilde{R}|_I^2]^{\frac{1}{2}}$, inspired from the solution to the optimal kinematic problem on $SO(3)$ [44] and the work by [35] on attitude tracking. We have introduced an arbitrary weighting matrix A as an additional tuning parameter and a small scalar ϵ that allows to remove the singularity at 180° while preserving the advantage of faster convergence rates obtained when using the gradient of the non-differentiable potential function Φ .

Another interesting choice for the state-dependent-gain function $k(\cdot)$ is the more aggressive function $k_3(\tilde{R}) = [1 + \epsilon - |\tilde{R}|_I^2]^{-1}$ for some small enough $\epsilon > 0$. The function $k_3(\tilde{R})$ satisfies the needed assumptions (positive, bounded and large for large \tilde{R}). This results in the following version of the nonlinear complementary filter on $SO(3)$

$$\text{Filter III} \quad \begin{cases} \dot{\hat{R}} &= \hat{R} [\omega_y]_{\times} - [\sigma]_{\times} \hat{R}, \\ \sigma_{k_3} &= -k_3(R_y \hat{R}^\top) \psi(A R_y \hat{R}^\top), \end{cases} \quad (40)$$

where $\hat{R} \in SO(3)$ is an estimate of R with $\hat{R}(0) = \hat{R}_0 \in SO(3)$ and A is a symmetric matrix such that $\bar{A} := \frac{1}{2}(\operatorname{tr}(A)I - A)$ is positive definite. Note that the attitude estimation scheme proposed in [34] can be recovered (in the bias-free case) from the above attitude estimator by taking $A = I$ and setting $\epsilon \rightarrow 0$. This innovation term (with $A = I$ and $\epsilon \rightarrow 0$) can be obtained from the gradient of the following *barrier-like* potential function $\Psi(\tilde{R}) = -\ln(1 - |\tilde{R}|_I^2)$.

However, the innovation term σ_{k_3} in (40) is not, as far as we know, a consequence of a gradient of any potential function but a novel design choice that has been introduced to obtain the desirable performance demonstrated in the following theorem.

Theorem 3. Consider the attitude kinematics system (15) coupled with the attitude observer (40) under assumption (18). Then, $\forall \epsilon, \gamma > 0$ such that $\gamma < [1 + \epsilon]^{-1}$, and for any $\tilde{R}(0) \in \mathcal{B}_{\xi_0}$ such that $\xi_0 = 1 - \gamma\epsilon/(1 - \gamma)$, one has

$$\underline{\beta}(|\tilde{R}(0)|_I, t) \leq |\tilde{R}(t)|_I \leq \bar{\beta}(|\tilde{R}(0)|_I, t), \quad (41)$$

for all $t \geq 0$, such that

$$\bar{\beta}(|\tilde{R}(0)|_I, t) = |\tilde{R}(0)|_I e^{-\gamma\lambda_{\min}^{\bar{A}} t},$$

$$\underline{\beta}(|\tilde{R}(0)|_I, t) = |\tilde{R}(0)|_I e^{-\lambda_{\max}^{\bar{A}} t}.$$

Proof. Let $\epsilon, \gamma > 0$ such that $\gamma \leq [1 + \epsilon]^{-1}$. Then, one can verify that the scalar $\xi_0 = 1 - \gamma\epsilon/(1 - \gamma)$ is non-negative. Therefore, the ball \mathcal{B}_{ξ_0} is well defined and non-empty. Let $\tilde{R}(0) \in \mathcal{B}_{\xi_0}$. Following similar steps as in (23)-(24) and in view of the fact that $\sigma_{k_3} = -\psi(A\tilde{R})/(1 + \epsilon - |\tilde{R}|_I^2)$, one obtains

$$\begin{aligned} -2\lambda_{\max}^{\bar{A}} \frac{1 - |\tilde{R}|_I^2}{1 + \epsilon - |\tilde{R}|_I^2} |\tilde{R}|_I^2 &\leq \frac{d}{dt} |\tilde{R}|_I^2 \\ &\leq -2\lambda_{\min}^{\bar{A}} \frac{1 - |\tilde{R}|_I^2}{1 + \epsilon - |\tilde{R}|_I^2} |\tilde{R}|_I^2. \end{aligned} \quad (42)$$

Therefore, the attitude error $|\tilde{R}(t)|_I^2$ is strictly decaying on \mathcal{B}_{ξ_0} which implies that \mathcal{B}_{ξ_0} is forward invariant and, hence, $\tilde{R}(t) \in \mathcal{B}_{\xi_0}$ for all $t \geq 0$. This implies that one has

$$0 \leq |\tilde{R}(t)|_I^2 < 1 - \gamma\epsilon/(1 - \gamma) < 1, \quad \forall t \geq 0,$$

which, after few algebraic manipulations, leads to

$$\gamma < \frac{1 - |\tilde{R}(t)|_I^2}{1 + \epsilon - |\tilde{R}(t)|_I^2} < 1, \quad \forall t \geq 0.$$

It follows from (42) that

$$-2\lambda_{\max}^{\bar{A}} |\tilde{R}|_I^2 \leq \frac{d}{dt} |\tilde{R}|_I^2 \leq -2\gamma\lambda_{\min}^{\bar{A}} |\tilde{R}|_I^2.$$

which yields the result of the theorem using the comparison lemma. \square

It should be mentioned that the three discussed filters above (Filter I, Filter II and Filter III) all share the same performance properties for small attitude errors (local performance). This is due to the fact that, for ϵ sufficiently small, one has the term $[1 + \epsilon - |\tilde{R}|_I^2] \rightarrow 1$ for small values of $|\tilde{R}|_I^2$. Consequently the innovation terms for the three filters become identical and hence the performance (convergence, filtering...etc). The difference between the three filters is remarkable as the attitude error increases. To illustrate this, we plot the variations of the norm of the innovation terms σ_{k_1} , σ_{k_2} and σ_{k_3} for all the three different filters. Let $\tilde{R} = \mathcal{R}_a(\theta, e_3)$ where $\theta \in [0, 2\pi]$ and $e_3 = [0, 0, 1]^T$. Consider a weighting matrix $A = \text{diag}([1, 2, 3])$ and let us choose different values for $\epsilon = 0.1, 0.01$ and 0.001 . It can be seen from Fig. 1a and Fig. 1b that the innovation term σ for all the three different filters is bounded (for a fixed ϵ) for

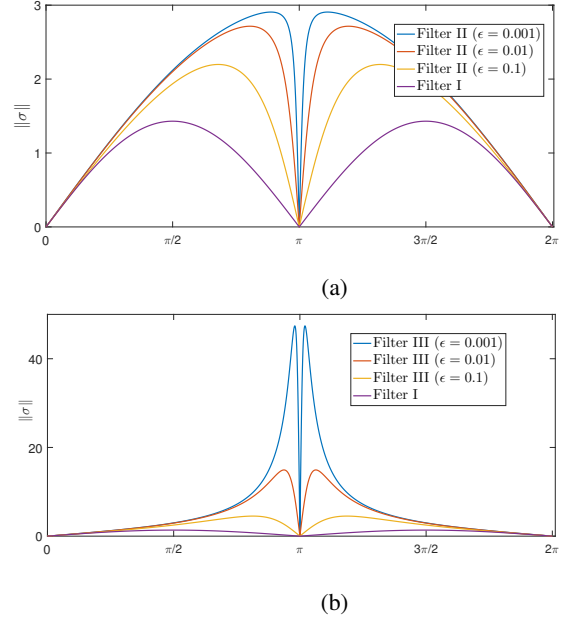


Fig. 1: Variations of the norm of the innovation term σ with respect to the attitude error angle θ : (a) Filter II (b) Filter III.

all attitude angles. Both Filter II and Filter III use a larger (in terms of norm) correction term when the attitude error is large compared to the traditional estimation scheme given by Filter I. As ϵ is chosen smaller, the correction term becomes larger for large attitude errors. Moreover, Filter III employs a more aggressive correction term compare to Filter II. This explains the larger picks in Fig. 1b. On the other hand, for all filters the σ term vanishes at attitudes of angle 180° around the eigen-axis e_3 which represents one of the undesired equilibria for the filters characterized by $\mathcal{R}_a(\pi, \mathcal{E}(A)) = \cup_{i=1,2,3} \mathcal{R}_a(\pi, e_i)$.

V. ROBUSTNESS ANALYSIS FOR DIFFERENT NONLINEAR COMPLEMENTARY FILTERS ON $SO(3)$ IN THE PRESENCE OF MEASUREMENT ERRORS

In this section we aim to study the robustness of the non-linear complimentary filters proposed in the previous section to the following measurement errors:

- Bounded errors in the angular velocity measurements such as noise, bias, disturbances...etc.
- Small errors in the attitude measurements.

As far as we know, robustness on the compact manifold $SO(3)$ has not been formulated before, at least in the context we study here. In fact, it is not clear how to define the meaning of *divergence* and *instability* which are necessary to justify the notion of “robustness”. In dynamical systems theory, a system is said to be unstable if at least one state variable in the system evolves without bounds (unbounded). The meaning of *unbounded* state is obviously related to the chosen *metric* (distance) on the given configuration space. For Euclidean spaces, for example, a state that evolves unbounded means that its Euclidean norm grows to infinity (∞); which represents the maximum distance that the metric allows.

However, the states of $SO(3)$ (rotation matrices) are naturally bounded with respect to any chosen smooth metric thanks to the geometry of the manifold. For our present purpose it is justified to relax the notion of instability on $SO(3)$ as follows:

Definition 2. *Given a Riemannian metric on $SO(3)$, a dynamical system on $SO(3)$ is said to be unstable if the state variable of the system (namely the attitude matrix $\tilde{R} \in SO(3)$) evolves to the manifold of maximum distance.*

According to Definition 3, let us choose the canonical Riemannian metric on $SO(3)$, called also the Euclidean metric. This metric results in the Euclidean distance defined in (2). For a given rotation matrix $\tilde{R} \in SO(3)$, the maximum distance $|\tilde{R}|_I = 1$ is obtained when $\tilde{R} \in \Pi$; manifold of all rotations of angle 180° . Interestingly, the Rodrigues vector $\mathcal{Z}(\tilde{R}) \in \mathbb{R}^3$ grows unbounded when $\tilde{R} \in \Pi$. This motivates to study the robustness of the dynamics of the Rodrigues vector $\mathcal{Z}(\tilde{R})$ with respect to measurement errors in the traditional sense of robustness on Euclidean spaces. More specifically, existing results on Input-to-State-Stability (ISS) on Euclidean spaces can be directly applied to the dynamics of the Rodrigues vector $\mathcal{Z}(\tilde{R})$ to derive conclusions about the robustness of the proposed filters' error dynamics.

A. Robustness Study to Gyro Measurement Errors

Here we assume that the gyro measurements are given according (16) for some bounded error vector n_ω . We also consider perfect attitude information such that $R_y \equiv R$. Following similar steps as in (22) and (25), it can be verified that the dynamics of the Rodrigues vector (for the three different discussed filters) in the presence of angular velocity measurement errors are given by the following differential equations:

$$\text{Filter I: } \dot{\mathcal{Z}}(\tilde{R}) = -k_1(\mathcal{Z}(\tilde{R}))\bar{A}\mathcal{Z}(\tilde{R}) - g(\mathcal{Z}(\tilde{R}))\hat{R}n_\omega, \quad (43)$$

$$\text{Filter II: } \dot{\mathcal{Z}}(\tilde{R}) = -k_2(\mathcal{Z}(\tilde{R}))\bar{A}\mathcal{Z}(\tilde{R}) - g(\mathcal{Z}(\tilde{R}))\hat{R}n_\omega, \quad (44)$$

$$\text{Filter III: } \dot{\mathcal{Z}}(\tilde{R}) = -k_3(\mathcal{Z}(\tilde{R}))\bar{A}\mathcal{Z}(\tilde{R}) - g(\mathcal{Z}(\tilde{R}))\hat{R}n_\omega, \quad (45)$$

where the scalar valued functions k_1, k_2 and k_3 are given by the following expressions

$$\begin{aligned} k_1(\mathcal{Z}(\tilde{R})) &= 1, \\ k_2(\mathcal{Z}(\tilde{R})) &= \left(\frac{1 + \|\mathcal{Z}(\tilde{R})\|^2}{1 + \epsilon(1 + \|\mathcal{Z}(\tilde{R})\|^2)} \right)^{\frac{1}{2}}, \\ k_3(\mathcal{Z}(\tilde{R})) &= \frac{1 + \|\mathcal{Z}(\tilde{R})\|^2}{1 + \epsilon(1 + \|\mathcal{Z}(\tilde{R})\|^2)}, \end{aligned}$$

and $g(\mathcal{Z}(\tilde{R})) = \frac{1}{2}(I + [\mathcal{Z}(\tilde{R})]_\times + \mathcal{Z}(\tilde{R})\mathcal{Z}(\tilde{R})^\top)$. Our goal in this subsection is to study the ISS property of the above dynamical systems with respect to bounded gyro disturbances n_ω . Before doing so important remarks are in order.

It can be noticed from (43)-(45) that, for small attitude estimation errors, the transfer function from the disturbance signal n_ω to the attitude vector $\mathcal{Z}(\tilde{R})$ satisfies

$$H(s) = \frac{1}{2}(sI + \bar{A})^{-1},$$

Note that all the proposed filters have the same transfer function $H(s)$ as above near the desired equilibrium point. The cutoff frequencies of $H(s)$ are given by

$$2\pi f_{c_i} = \lambda_i^{\bar{A}},$$

where $\lambda_i^{\bar{A}}$ denotes the i -th eigenvalue of \bar{A} . Therefore, as the magnitude of the eigenvalues of \bar{A} decreases the cutoff frequency of the filters decreases and high frequency components of the disturbance signal are filtered. On the other hand, the eigenvalues of \bar{A} directly affect the speed of convergence of the filters as well. Larger values of $\lambda_i^{\bar{A}}$ means faster convergence rates. This shows the trade-off that one would expect when considering the traditional nonlinear complementary filter (19) (constant gain). The attitude filters proposed in (35) and (40), however, solve this conflict between the speed of convergence and the cutoff frequency by considering a state-dependent time-varying gain which increases as the attitude estimation error grows and decreases to 1 for small attitude estimation errors. Therefore, one may pick up small values of $\lambda_i^{\bar{A}}$ to guarantee better filtering while still maintaining a good convergence speed when using the attitude filters proposed in this paper.

Now let us study rigorously the ISS property of the proposed attitude filtering schemes. First, let us prove the following interesting "existence" result that motivates the need to carefully investigate the robustness of the nonlinear complementary filter on $SO(3)$ discussed in the previous section.

Proposition 1. *Consider the dynamics (43) obtained from the error dynamics of Filter I. Assume that there exists $i \in \{1, 2, 3\}$ such that*

$$\begin{aligned} \hat{R}(t)n_\omega(t) &= -2\lambda_i^{\bar{A}}\mathcal{Z}(\tilde{R}(0))(2\lambda_i^{\bar{A}}t + 1)^{-\frac{1}{2}}, \\ \bar{A}\mathcal{Z}(\tilde{R}(0)) &= \lambda_i^{\bar{A}}\mathcal{Z}(\tilde{R}(0)), \quad \|\mathcal{Z}(\tilde{R}(0))\| = 1. \end{aligned}$$

Then $\mathcal{Z}(\tilde{R}(t)) = \mathcal{Z}(\tilde{R}(0))(2\lambda_i^{\bar{A}}t + 1)^{\frac{1}{2}}$ for all $t \geq 0$.

Proof. First we show that, under the conditions of Proposition 1, the direction of the Rodrigues vector $\mathcal{Z}(\tilde{R}(t))$ remains constant for all times. In view of (43), one has

$$\begin{aligned} \frac{d}{dt}\|\mathcal{Z}(\tilde{R})\|^2 &= -2\mathcal{Z}(\tilde{R})^\top \bar{A}\mathcal{Z}(\tilde{R}) - \\ &\quad (1 + \|\mathcal{Z}(\tilde{R})\|^2)\mathcal{Z}(\tilde{R})^\top \hat{R}n_\omega. \end{aligned} \quad (46)$$

Therefore, using (43) and (46), it can be shown that $\zeta := \frac{\mathcal{Z}(\tilde{R})}{\|\mathcal{Z}(\tilde{R})\|}$ satisfies the following differential equation

$$\frac{d}{dt}\zeta = [\zeta]_\times^2 \bar{A}\zeta - \frac{1}{2}([\zeta]_\times - \frac{[\zeta]_\times^2}{\|\mathcal{Z}(\tilde{R})\|})\hat{R}n_\omega. \quad (47)$$

Hence, it is straightforward to see that, under the assumption of the proposition, one has

$$\frac{d}{dt}\zeta(0) = 0,$$

which implies that the initial direction is an equilibrium point of the non-autonomous dynamics (47) and, hence, the direction of the Rodrigues vector $\mathcal{Z}(\tilde{R}(t))$ is constant for all $t \geq 0$. Therefore, the angle between the two signals $\mathcal{Z}(\tilde{R}(t))$ and $\hat{R}(t)n_\omega(t)$ remains zero for all times. It follows, from equation (46) that

$$\begin{aligned} \frac{d}{dt}\|\mathcal{Z}(\tilde{R})\|^2 &= -2\lambda_i^{\bar{A}}\|\mathcal{Z}(\tilde{R})\|^2 + \\ &\quad 2\lambda_i^{\bar{A}}(1 + \|\mathcal{Z}(\tilde{R})\|^2)\|\mathcal{Z}(\tilde{R})\|(2\lambda_i^{\bar{A}}t + 1)^{-\frac{1}{2}}. \end{aligned}$$

It can be checked, by direct substitution in the above equation, that $\|\mathcal{Z}(\tilde{R}(t))\| = (2\lambda_i^A t + 1)^{\frac{1}{2}}$ is a solution. This proves the proposition. \square

In Proposition 1, it is shown that the traditional gradient-based nonlinear attitude observer on $SO(3)$ is not ISS with respect to bounded angular velocity measurements disturbances. In fact, one can construct a bounded and vanishing disturbance that prevents the observer from converging to the actual attitude. If we consider a particular time-dependent vanishing measurement disturbance $\tilde{R}n_\omega(t)$ with an initial attitude error $|\tilde{R}(0)| = 1/\sqrt{2}$, or equivalently $\|\mathcal{Z}(\tilde{R}(0))\| = 1$, which corresponds to an angle of rotation equals 90° , the attitude error tends to the undesired manifold, i.e., $|\tilde{R}|_I \rightarrow 1$ ($\|\mathcal{Z}(\tilde{R})\| \rightarrow \infty$) where the error angle is maximum and equals 180° . It should be mentioned that the result of Proposition (1) does not intend to question the applicability of the nonlinear complimentary filter of [14]. Nevertheless, the discussions of this section have motivated us to think more rigorously about the issue of robustness and convergence speed when designing attitude observers on $SO(3)$.

Theorem 4. *System (43) (resp. (44) and (45)) is locally input-to-state stable. In particular, for all $r > 0$ and $0 < \varrho < 1$, there exists β_1 (resp. β_2 and β_3) $\in \mathcal{KL}$ such that for all $\|\mathcal{Z}(\tilde{R}(0))\| < r$ and $\sup_{t \geq 0} \|n_\omega(t)\| < k_{u1}$ (resp. k_{u2} and k_{u3}), one has*

$$\|\mathcal{Z}(\tilde{R}(t))\| \leq \beta_i(\|\mathcal{Z}(\tilde{R}(0))\|, t) + \gamma_i(\sup_{t \geq 0} \|n_\omega(t)\|),$$

for $i = 1, 2, 3$ with $k_{ui} = \gamma_i^{-1}(r)$, $\gamma_i(s) = \varsigma_i(r)s/2\varrho\lambda_{\min}^A$ and

$$\begin{aligned} \varsigma_1(r) &= (1 + r^2), \\ \varsigma_2(r) &= ((1 + r^2)(1 + \epsilon + \epsilon r^2))^{\frac{1}{2}}, \\ \varsigma_3(r) &= 1 + \epsilon + \epsilon r^2. \end{aligned}$$

Before addressing the proof of Theorem 4 some remarks are in order. Theorem 4 shows that all the attitude filters (Filter I, Filter II and Filter III) discussed in this paper are Locally Input-to-State Stable (LISS) in the sense of Definition 1. It should be mentioned that the conclusion of LISS can be directly inferred from the fact that the unforced ($n_\omega \equiv 0$) systems (43)-(45) are globally asymptotically stable by using the result of [38, Lemma I.1]. However, the conclusions of Theorem 4 give “explicitly” the bounds k_{ui} , $i = 1, 2, 3$ on the disturbance n_ω where LISS holds for all the proposed filters. The result of Theorem 4 allows us to compare rigorously the robustness of these filters to gyro measurement errors.

For a given constant $r > 0$ in Theorem 4, the explicit bounds on $\sup_{t \geq 0} \|n_\omega(t)\|$ can be derived as

$$\begin{aligned} k_{u1} &= \frac{\varrho\lambda_{\min}^A r}{1 + r^2}, \\ k_{u2} &= \frac{\varrho\lambda_{\min}^A r}{((1 + r^2)(1 + \epsilon + \epsilon r^2))^{\frac{1}{2}}}, \\ k_{u3} &= \frac{\varrho\lambda_{\min}^A r}{1 + \epsilon + \epsilon r^2}. \end{aligned}$$

The constant $r > 0$, in Theorem 4, can be arbitrarily large to cover all initial conditions for the attitude error $\tilde{R} \in SO(3)$.

For Filter I, as r gets larger the value of k_{u1} , which corresponds to the bound on the allowed disturbances, gets smaller. This fact suggests that as we start closer to large attitude errors the robustness to small measurement gyro disturbances may be lost. In contrast, for Filter III for example, for any large $r > 0$ we can always choose the parameter ϵ small enough to such that k_{u3} is also large and therefore the allowed bound on the gyro disturbances are much larger. To put this together, by setting ϵ small enough such that $\epsilon < r^2/(1 + r^2)$, it can be verified that

$$k_{u1} < k_{u2} < k_{u3}.$$

Consequently, it can be concluded that Filter III has the best robustness to gyro measurement errors while Filter I exhibits a reduced robustness compared to the other two proposed filters. Moreover, by letting $\epsilon \rightarrow 0$ and $r \rightarrow \infty$, it can be noticed that $k_{u1} \rightarrow 0$, $k_{u2} \rightarrow \varrho\lambda_{\min}^A$ and $k_{u3} \rightarrow \infty$ which (in this case) leads to conclude that Filter I is not ISS, Filter II is ISS with respect to all disturbances such that $\sup_{t \geq 0} \|n_\omega(t)\| < \varrho\lambda_{\min}^A$ and Filter III has the Global ISS property.

Proof of Theorem 4. Consider the following ISS-Lyapunov functions candidate

$$V_i(\mathcal{Z}(\tilde{R})) = \frac{1}{2} \|\mathcal{Z}(\tilde{R}(t))\|^2, \quad i = 1, 2, 3.$$

The time derivative of V_1 (resp. V_2 and V_3) along the trajectories of (43) (resp. (44) and (45)) satisfies

$$\begin{aligned} \dot{V}_i(\mathcal{Z}(\tilde{R})) &= -k_i(\mathcal{Z}(\tilde{R}))\mathcal{Z}(\tilde{R})^\top \tilde{A}\mathcal{Z}(\tilde{R}) - \\ &\quad \frac{1}{2}(1 + \|\mathcal{Z}(\tilde{R})\|^2)\mathcal{Z}(\tilde{R})^\top \tilde{R}n_\omega \\ &\leq -\lambda_{\min}^A(1 - \varrho)k_i(\mathcal{Z}(\tilde{R}))\|\mathcal{Z}(\tilde{R})\|^2 + \\ &\quad \frac{1}{2}\|\mathcal{Z}(\tilde{R})\|(1 + \|\mathcal{Z}(\tilde{R})\|^2) \\ &\quad \left(\|n_\omega\| - \frac{2k_i(\mathcal{Z}(\tilde{R}))\varrho\lambda_{\min}^A\|\mathcal{Z}(\tilde{R})\|}{1 + \|\mathcal{Z}(\tilde{R})\|^2} \right), \end{aligned}$$

for $i = 1, 2, 3$. Assume that $\|\mathcal{Z}(\tilde{R})\| \leq r$. Then, it is clear that for all $\|\mathcal{Z}(\tilde{R})\| \geq \rho(\|n_\omega\|)$, with $\rho(s) = \gamma(s) = (1 + r^2)s/2k_i(r)\varrho\lambda_{\min}^A$, one has

$$\dot{V}_i(\mathcal{Z}(\tilde{R})) \leq -\lambda_{\min}^A(1 - \varrho)k_i(\mathcal{Z}(\tilde{R}))\|\mathcal{Z}(\tilde{R})\|^2.$$

Applying the result of Lemma 2 concludes the proof. \square

Another interesting feature of the attitude estimation scheme given by Filter III is demonstrated in the following theorem when the tuning scalar $\epsilon \rightarrow 0$.

Theorem 5. *Consider the attitude kinematics system (15) coupled with the attitude observer (40). Assume that $\tilde{R}(0) \in SO(3) \setminus \Pi$. Then the innovation term σ in (40), with $A = aI > 0$ and $\epsilon = 0$, minimizes the following cost functional*

$$\begin{aligned} J(\sigma) &= \sup_{n_\omega \in \mathcal{N}} \left\{ \lim_{t \rightarrow +\infty} \left[2\ln(1 + \|\mathcal{Z}(\tilde{R}(t))\|^2) + \right. \right. \\ &\quad \left. \left. \int_0^t \left((2a - \frac{1}{\gamma^2})\mathcal{Z}(\tilde{R})^\top \mathcal{Z}(\tilde{R}) + \frac{1}{2a}\sigma^\top \sigma - \gamma^2 n_\omega^\top n_\omega \right) d\tau \right] \right\}, \end{aligned} \quad (48)$$

where $\gamma^2 > \frac{1}{2a}$ and \mathcal{N} is the set of locally bounded disturbances, with a value function $J^* = 2\ln(1 + \|\mathcal{Z}(\tilde{R}(0))\|^2)$. Moreover, the achieved disturbance attenuation level is

$$(4a - \frac{1}{\gamma^2}) \int_0^\infty \mathcal{Z}(\tilde{R}(t))^\top \mathcal{Z}(\tilde{R}(t)) dt \leq \gamma^2 \int_0^\infty n_\omega(t)^\top n_\omega(t) dt + 2\ln(1 + \|\mathcal{Z}(\tilde{R}(0))\|^2). \quad (49)$$

Proof. Recall from (6) and (22) that the dynamics of the Rodrigues vector are written as

$$\begin{aligned} \frac{d}{dt} \mathcal{Z}(\tilde{R}) &= \frac{1}{2} (I + [\mathcal{Z}(\tilde{R})]_\times + \mathcal{Z}(\tilde{R}) \mathcal{Z}(\tilde{R})^\top) (\sigma - \hat{R} n_\omega) \\ &:= g(\mathcal{Z}(\tilde{R})) (\sigma - \hat{R} n_\omega). \end{aligned}$$

Consider the following Lyapunov function candidate

$$V(\mathcal{Z}(\tilde{R})) = \frac{1}{2} \ln(1 + \|\mathcal{Z}(\tilde{R})\|^2). \quad (50)$$

The Lie Derivative of V along g satisfies

$$\begin{aligned} L_g V(\mathcal{Z}(\tilde{R})) &= \nabla V(\mathcal{Z}(\tilde{R})) g(\mathcal{Z}(\tilde{R})) \\ &= \frac{\mathcal{Z}(\tilde{R})^\top}{1 + \|\mathcal{Z}(\tilde{R})\|^2} g(\mathcal{Z}(\tilde{R})) \\ &= \frac{1}{2} \mathcal{Z}(\tilde{R})^\top. \end{aligned}$$

Let $W_1 = \gamma^2 I$ and consider the following auxiliary system

$$\frac{d}{dt} \mathcal{Z}(\tilde{R}) = W_1^{-1} g(\mathcal{Z}(\tilde{R})) (L_g V(\mathcal{Z}(\tilde{R})))^\top + g(\mathcal{Z}(\tilde{R})) \sigma. \quad (51)$$

with $\sigma = \frac{1}{2} \alpha(\mathcal{Z}(\tilde{R})) := -W_2^{-1} (L_g V(\mathcal{Z}(\tilde{R})))^\top = -a \mathcal{Z}(\tilde{R})$, where $W_2 = \frac{1}{2a} I$. Then, the auxiliary system (51) becomes

$$\frac{d}{dt} \mathcal{Z}(\tilde{R}) = -\frac{2a - \frac{1}{\gamma^2}}{4} (1 + \|\mathcal{Z}(\tilde{R})\|^2) \mathcal{Z}(\tilde{R}), \quad (52)$$

which is clearly globally asymptotically stable as long as the scalar $2a - \frac{1}{\gamma^2}$ is strictly positive. Consequently, using the result of [45, Theorem 5.1], it follows that

$$\sigma = \alpha(\mathcal{Z}(\tilde{R})) = -2a \mathcal{Z}(\tilde{R}) = -a \frac{\psi(\tilde{R})}{1 - |\tilde{R}|_2^2}$$

solves the inverse optimal \mathcal{H}_∞ problem by minimizing the cost functional

$$J(\sigma) = \sup_{n_\omega \in \mathcal{N}} \left\{ \lim_{t \rightarrow +\infty} \left[2\ln(1 + \|\mathcal{Z}(\tilde{R}(t))\|^2) + \int_0^t (l(\mathcal{Z}(\tilde{R})) + \sigma^\top W_2 \sigma - n_\omega^\top W_1 n_\omega) d\tau \right] \right\}, \quad (53)$$

where

$$\begin{aligned} l(x) &= -4(L_g V(x) W_1^{-1} L_g V(x)^\top - L_g V(x) W_2^{-1} L_g V(x)^\top) \\ &= (2a - \frac{1}{\gamma^2}) x^\top x. \end{aligned}$$

Substituting $\sigma = \alpha(\mathcal{Z}(\tilde{R}))$ in (48) and using the fact that $J(\sigma) \leq J^*$ it follows that

$$\begin{aligned} \int_0^\infty ((4a - \frac{1}{\gamma^2}) \mathcal{Z}(\tilde{R})^\top \mathcal{Z}(\tilde{R}) - \gamma^2 n_\omega^\top n_\omega) dt &\leq \\ J^* &= 2\ln(1 + \|\mathcal{Z}(\tilde{R}(0))\|^2), \end{aligned} \quad (54)$$

which proves the result. \square

Theorem 5 shows that the choice of the observer innovation term σ in (40) in the ideal case where $\epsilon \rightarrow 0$ solves a meaningful inverse optimal \mathcal{H}_∞ optimization problem. Moreover, a bound on the disturbance attenuation level is obtained. This result leads naturally to conclude on the ISS-type robustness of the attitude estimation scheme (40), when $A = aI$ and $\epsilon = 0$.

B. Robustness Study to Attitude Errors

In this subsection, we assume that the attitude information R_y is obtained according to (17) for some *small* perturbation attitude matrix $N_R \in SO(3)$. We also consider perfect gyro measurements such that $\omega_y \equiv \omega$. This allows us to study the two robustness problems separately.

The new “available” attitude error is given by $\tilde{R}_y = R_y \hat{R}^\top = N_R \tilde{R}$. The contaminated attitude error \tilde{R}_y will be used in the innovation term σ in (19), (35) and (40) for the three different versions of the nonlinear complimentary filter. For simplicity of discussions, we consider in this work only the case where N_R is a perturbation rotation of small angle in the direction of the rotation \tilde{R} . Explicitly, if the orientation \tilde{R} is described by $\mathcal{R}_a(\theta, u)$ for some $\theta \in \mathbb{R}$ and $u \in \mathbb{S}^2$ then we consider $N_R = \mathcal{R}_a(n_\theta, u)$ for some small $n_\theta \ll 1$. This implies that the available attitude error satisfies $\tilde{R}_y = \mathcal{R}_a((\theta + n_\theta), u)$. Using the fact that $\mathcal{Z}(\mathcal{R}_a(x, v)) = \tan(x/2)v$ for all $x \in \mathbb{R}$ and $v \in \mathbb{S}^2$, it can be verified that

$$\begin{aligned} \mathcal{Z}(\tilde{R}_y) &= \frac{\tan(\theta/2) + \tan(n_\theta/2)}{1 - \tan(\theta/2)\tan(n_\theta/2)} u \\ &\simeq \frac{\tan(\theta/2) + n_\theta/2}{1 - \tan(\theta/2)n_\theta/2} u \\ &= \mathcal{Z}(\tilde{R}) + \frac{n_\theta(1 + \|\mathcal{Z}(\tilde{R})\|^2)}{2 - n_\theta \|\mathcal{Z}(\tilde{R})\|} u, \end{aligned}$$

where we have used the following first order approximations $\tan(x) \simeq x$ for all small enough $x \in \mathbb{R}$. Now, we need to re-evaluate the expression of the innovation term σ in terms of $\mathcal{Z}(\tilde{R}_y)$. In view of (10), the expression of σ for the three filters is given by

$$\sigma = -2k_i(\mathcal{Z}(\tilde{R}_y)) \frac{(I - [\mathcal{Z}(\tilde{R}_y)]_\times)}{1 + \|\mathcal{Z}(\tilde{R}_y)\|^2} \bar{A} \mathcal{Z}(\tilde{R}_y), \quad i = 1, 2, 3.$$

Moreover, one has

$$1 + \|\mathcal{Z}(\tilde{R}_y)\|^2 = \frac{(n_\theta^2 + 4)(1 + \|\mathcal{Z}(\tilde{R})\|^2)}{(2 - n_\theta \|\mathcal{Z}(\tilde{R})\|)^2} \simeq \frac{4(1 + \|\mathcal{Z}(\tilde{R})\|^2)}{(2 - n_\theta \|\mathcal{Z}(\tilde{R})\|)^2},$$

where the second order term in n_θ^2 was neglected ($n_\theta \ll 1$). On the other hand, recall that the dynamics of $\mathcal{Z}(\tilde{R})$ satisfies $\dot{\mathcal{Z}}(\tilde{R}) = g(\mathcal{Z}(\tilde{R}))\sigma$ which implies that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathcal{Z}(\tilde{R})\|^2 \\ &= \mathcal{Z}(\tilde{R})^\top g(\mathcal{Z}(\tilde{R}))\sigma \\ &= \frac{1}{2} (1 + \|\mathcal{Z}(\tilde{R})\|^2) \mathcal{Z}(\tilde{R})^\top \sigma \\ &= -\frac{k_i(\mathcal{Z}(\tilde{R}_y))}{4} \mathcal{Z}(\tilde{R})^\top \tilde{A} \mathcal{Z}(\tilde{R}_y) (2 - n_\theta \|\mathcal{Z}(\tilde{R})\|^2) \\ &\lesssim -k_i(\mathcal{Z}(\tilde{R}_y)) \lambda_{\min}^{\tilde{A}} \|\mathcal{Z}(\tilde{R})\|^2 - \\ &\quad \frac{1}{2} k_i(\mathcal{Z}(\tilde{R}_y)) \lambda_{\min}^{\tilde{A}} n_\theta \|\mathcal{Z}(\tilde{R})\| (1 - \|\mathcal{Z}(\tilde{R})\|^2), \end{aligned}$$

where again higher order terms in n_θ were neglected. Therefore, one concludes that

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{Z}(\tilde{R})\|^2 \leq -\lambda_{\min}^{\tilde{A}} (1 - \varrho) k_i(\mathcal{Z}(\tilde{R}_y)) \|\mathcal{Z}(\tilde{R})\|^2,$$

for all $\|\mathcal{Z}(\tilde{R})\| \geq \rho(|n_\theta|)$ such that $\|\mathcal{Z}(\tilde{R}(0))\| < r$ and $\rho(s) = (1 + r^2)s/2$. Therefore, according to Lemma 2, the dynamics of $\mathcal{Z}(\tilde{R})$ (for all three filters) are LISS for all $\|\mathcal{Z}(\tilde{R}(0))\| < r$ and $\sup_{t \geq 0} |n_\theta(t)| < 2r/(1 + r^2)$. As it is noticed, the upper bound on the allowed attitude measurements $n_\theta(t)$ decreases as the initial condition gets larger. This results can be intuitively explained by the fact that, for large attitude errors close enough to 180° , the attitude noise can *mislead* the innovation term $\psi(A\tilde{R})$ to change the direction of the correction and therefore a correction is applied in the wrong direction which causes the attitude to get closer to 180° . If we are unlucky enough, small noise can cause chattering near the undesired manifold of all rotations of angle 180° . The reader is referred to some recent works on hybrid observers on $SO(3)$ where hysteresis-like switching mechanisms have been proposed to guarantee global stability results with robustness to small measurements noise [29]–[31].

VI. VECTOR MEASUREMENTS FORMULATION OF THE PROPOSED COMPLEMENTARY FILTERS ON $SO(3)$

The nonlinear complementary filters discussed in the previous sections were written in terms of the attitude information $R_y(t)$ which is not available, in practice, directly using any sensor. However, body-frame measurements of constant known inertial vectors can be obtained using different sensors such as accelerometers, magnetometers, star trackers, cameras...etc. We assume we have $n \geq 2$ vector measurements

$$b_i = R^\top r_i, \quad i = 1, \dots, n,$$

where r_i are some known constant inertial vectors. Moreover, we assume that at least two vector measurements b_i are non-collinear. This is a standard assumption in attitude estimation which is necessary to recover the full attitude information from the available data. To implement one of the discussed attitude filters on $SO(3)$ in practice, we need to reconstruct the attitude matrix R_y using some static attitude determination algorithms such that $R_y = f_{\text{reconst}}((b_i, r_i)_{1 \leq i \leq n})$. Obviously if the measurements b_i are perfect then the reconstruction gives

perfect attitude such that $R_y \equiv R$. However, this is not realistic as the noise in the vector measurements b_i is probably to propagate to R_y . Attitude reconstruction schemes are likely to be sensitive to noise which motivates [14] to explicitly formulate the traditional nonlinear complementary filter using directly available measurements b_i .

To do so, we use the results from [28, Proposition 5] to derive the following identities

$$\psi(AR\hat{R}^\top) = \frac{1}{2} \hat{R} \sum_{i=1}^n \rho_i (b_i \times \hat{R}^\top r_i), \quad (55)$$

where $A = \sum_{i=1}^n \rho_i r_i r_i^\top$ such that $\rho_i, i = 1, \dots, n$ are positive scalars. Note that under the assumption that two vectors b_1 and b_2 are noncollinear, one guarantees that the positive semidefinite matrix A has rank greater or equal 2. Therefore, the matrix $\tilde{A} = \frac{1}{2}(\text{tr}(A) - A)$ can be shown to have full rank (positive definite) which allows to use it in (19), (35) and (40). It remains to express the norm $|\tilde{R}|_I^2$ which appears in (35) and (40) in the expression of the state-dependent gains.

Let b_1 and b_2 be two (non-collinear) body-frame vector measurements corresponding to the inertial unit vectors r_1 and r_2 such that $b_1 = R^\top r_1$ and $b_2 = R^\top r_2$. Let us define the vectors $u_1 = r_1/\|r_1\|$, $u_2 = (r_1 \times r_2)/\|r_1 \times r_2\|$ and $u_3 = u_1 \times u_2$ along with their corresponding body-frame vectors $w_1 = b_1/\|b_1\|$, $w_2 = (b_1 \times b_2)/\|b_1 \times b_2\|$ and $w_3 = w_1 \times w_2$. Then, one can verify that

$$|\tilde{R}|_I^2 = \frac{1}{8} \sum_{i=1}^3 \|w_i - \hat{R}^\top u_i\|^2, \quad (56)$$

which is a quite convenient formula for the computation of $k_2(\tilde{R})$ and $k_3(\tilde{R})$ in (35) and (40). Consequently, the results in (55) and (56) allow to write the proposed attitude filters in (19), (35) and (40) explicitly in terms of vector measurements without the need to reconstruct the attitude matrix R_y .

VII. IMPLEMENTATION ASPECTS AND NUMERICAL RESULTS

This section presents numerical examples and comparisons among the nonlinear complimentary attitude filters discussed in this paper. First, we derive the discrete-version of the nonlinear complementary filter on $SO(3)$ for practical implementation purposes. The class of nonlinear complementary filters on $SO(3)$ discussed in this paper can be written, in the continuous setting, as

$$\dot{\hat{R}}(t) = \hat{R}(t)[\hat{\omega}(t)]_\times, \quad \hat{R}(0) \in SO(3), \quad (57)$$

where estimated angular velocity $\hat{\omega}$ is given by

$$\hat{\omega}(t) = \omega_y(t) - \hat{R}^\top(t)\sigma(t)$$

and $\sigma(t) = -k(R_y(t)\hat{R}^\top(t))\psi(AR_y(t)\hat{R}^\top(t))$ such that $k(\cdot)$ depends on the type of filter used (Filter I, Filter II and Filter III). Assume that during the time interval $[t_k, t_{k+1})$, where $k \in \mathbb{N}$ and $t_0 = 0$, the estimated angular velocity $\hat{\omega}(t)$. This is a realistic assumption for small integration step sizes. Consequently in view of (57) it follows that

$$\frac{d}{dt} \left(\hat{R} e^{[\hat{\omega}(t_k)t]_\times} \right) = 0, \quad t \in [t_k, t_{k+1}).$$

Exact integration of the above equation between t_k and t_{k+1} yields the following update step on $SO(3)$

$$\hat{R}(t_{k+1}) = \hat{R}(t_k) e^{[\hat{\omega}(t_k)(t_{k+1}-t_k)]_{\times}}, \quad k \in \mathbb{N}. \quad (58)$$

Note that the exponential map on $SO(3)$ has a compact formula for quick computation (instead of using high order Taylor series) given by the map \mathcal{R}_a in (3) such that $e^{[x]_{\times}} = \mathcal{R}_a(\|x\|, x/\|x\|)$ for all $x \in \mathbb{R}^3$. Moreover, it is worth pointing out that the discrete integration rule (58) can be lifted to the quaternion space (using the quaternion multiplication rule) to simplify the computations. The resulting integration scheme can be verified to be equivalent to the discrete quaternion integration proposed in [46].

Consider the kinematics of the attitude system (15) with the following angular velocity input signal

$$\omega(t) = \begin{bmatrix} \sin(0.3t) \\ 0.7 \sin(0.2t + \pi) \\ 0.5 \sin(0.1t + \pi/3) \end{bmatrix} \text{ (rad/s),}$$

and initial condition $R(0) = I$. We implement a simulation of the real kinematic system (15) through the integration scheme on $SO(3)$ above using a high sampling rate of 1000 Hz. We assume that the gyro measurements of the angular velocity are obtained at 200 Hz and are contaminated by a white noise with zero mean and standard standard deviation equals 0.1(rad/s), see Fig. 2. We also consider body-frame measurements b_1

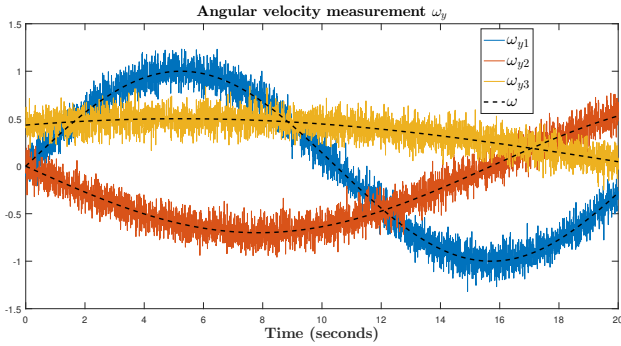


Fig. 2

and b_2 (same sampling frequency of 200 Hz) of two non-collinear inertial vectors given by $r_1 = [1, -1, 1]^T / \sqrt{3}$ and $r_2 = [0, 0, 1]^T$. We also consider additional white noise with zero mean and standard deviation equals 0.1 for both vector measurements b_1 and b_2 . All attitude errors are initialized at an attitude $\hat{R}(0) = \mathcal{R}_a(\pi - 10^{-1}, [1, 0, 0]^T)$. The two vectors r_1 and r_2 are weighted with gains $\rho_1 = 1$ and $\rho_2 = 2$, respectively. Therefore, the corresponding weighting matrix is given by

$$A = \rho_1 r_1 r_1^T + \rho_2 r_2 r_2^T = \frac{1}{3} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 7 \end{bmatrix}.$$

The formulas (55)-(56) are used to explicitly express the innovation term σ for the three different filters (Filter I, Filter II and Filter III). The parameter ϵ used in Filter II and Filter III innovation term is chosen small and equals $\epsilon = 10^{-2}$. The updated attitude estimates \hat{R} are obtained at

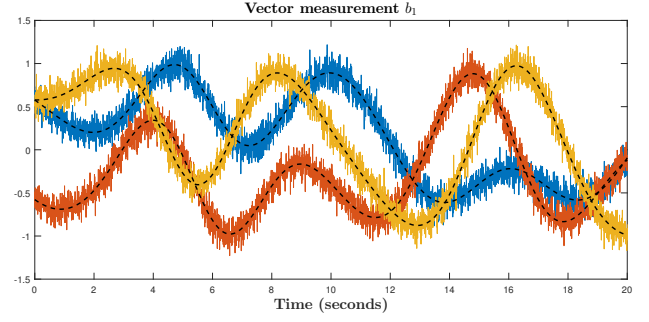


Fig. 3

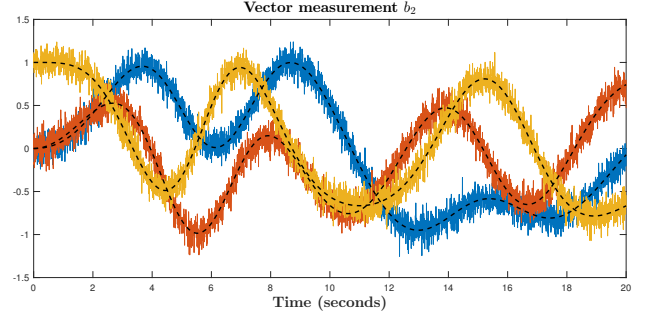


Fig. 4

a frequency of 200 Hz which corresponds to the frequency of the measurements. Attitude estimates norms for the three

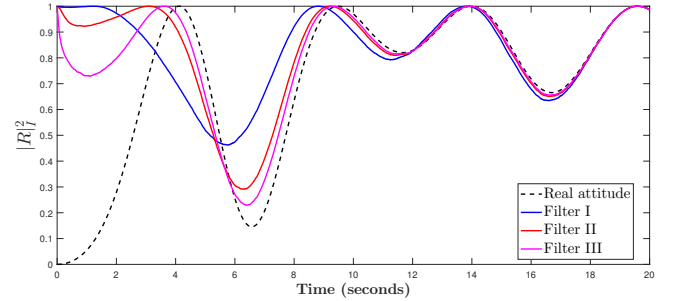


Fig. 5: Attitude estimates norm versus time for Filter I, Filter II and Filter III.

discussed filters are plotted in Fig. 5. As expected, Filter II and Filter III behave better than the constant gain filter (Filter I) in terms of convergence rate. Especially Filter III is able to correct its attitude in a faster time compared to the two other filters. Note that the three filters have an identical behaviour near the origin of attitude error so no performance is lost (locally) when introducing the state dependent gain filters. The innovation term σ for the three filters is plotted in Fig. 6. It can be seen that Filter III innovation term is very aggressive at initial times compared to the two other filters. Although this allows fast correction of the attitude. Filter II has a relatively less aggressive (compared to Filter III) correction while maintaining a considerably good speed (compared to Filter I).

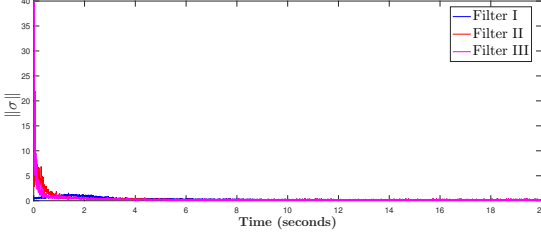


Fig. 6: Innovation term σ versus time.

VIII. CONCLUSION

The traditional nonlinear complementary filter on $SO(3)$ has been revisited and an explicit time-solution of the resulting attitude estimation error has been provided in the bias-free case. Almost global asymptotic (and local exponential) stability properties of this filter, that are usually determined using complex proofs, are easily deduced from the obtained closed form solution. The robustness of this filter has also been investigated and it has been shown that this filter is not ISS with respect to gyro measurement disturbances. As an alternative solution, we consider two nonlinear complementary filters (with state-dependent gains) and provide explicit solutions for the resulting estimation error. It is shown that these proposed observers lead to better results in terms of convergence and robustness to measurement errors. The domain of Local ISS (with respect to gyro errors) for the three discussed nonlinear complementary filters on $SO(3)$ is explicitly computed and reveals that the state dependent gain filters have a larger robustness domain with respect to angular velocity measurement errors. On the other hand, the robustness domain for the three filters with respect to small attitude measurement errors is shown to be the same for the three filters.

APPENDIX A PROOF OF LEMMA 1

Let $R \in SO(3)$ be an attitude matrix represented by a rotation of angle θ around some unit vector $u \in \mathbb{S}^2$. Using (3) and the fact that $[u]_{\times}^2 = -u^{\top}uI + uu^{\top}$, one can show that

$$|R|_I^2 = \frac{1}{4}\text{tr}(I - R) = \frac{1}{2}(1 - \cos(\theta)) = \sin^2(\theta/2).$$

On the other hand, the rotation matrix R^2 represents a rotation of the same direction u as R and with twice the angle of rotation of R . Hence, one has $|R^2|_I^2 = \sin^2(\theta) = 4\cos^2(\theta/2)\sin^2(\theta/2) = 4(1 - |R|_I^2)|R|_I^2$. Moreover, one has

$$\|\psi(R)\|^2 = -\frac{1}{2}\text{tr}(\mathbb{P}_a(R), \mathbb{P}_a(R)) = \text{tr}(I - R^2)/4 = |R^2|_I^2,$$

for all $R \in SO(3)$. On the other hand, using the fact that $\text{tr}(A[u]_{\times}) = 0$ for any symmetric matrix A and $u \in \mathbb{R}^3$ and $[u]_{\times}^2 = -u^{\top}uI + uu^{\top}$, one obtains

$$\begin{aligned} \text{tr}(A(I - R)) &= -\text{tr}(A(\sin(\theta)[u]_{\times} + (1 - \cos(\theta))[u]_{\times}^2)), \\ &= -(1 - \cos(\theta))\text{tr}(A[u]_{\times}^2), \\ &= (1 - \cos(\theta))u^{\top}\bar{A}u, \\ &= 2|R|_I^2u^{\top}\bar{A}u. \end{aligned}$$

Moreover, one has $\lambda_{\max}^{\bar{A}}\|u\|^2 \leq u^{\top}\bar{A}u \leq \lambda_{\max}^{\bar{A}}\|u\|^2$ and $\|u\|^2 = 1$ which proves (8).

Let $R \in SO(3) \setminus \Pi$ and hence $R = \mathcal{R}_r(\mathcal{Z}(R))$. In view of (4) one has

$$\begin{aligned} \mathbb{P}_a(AR) &= \frac{1}{2}(AR - R^{\top}A) \\ &= \frac{1}{1 + \|\mathcal{Z}(R)\|^2}(A\mathcal{Z}(R)\mathcal{Z}(R)^{\top} - \mathcal{Z}(R)\mathcal{Z}(R)^{\top}A \\ &\quad + A[\mathcal{Z}(R)]_{\times} + [\mathcal{Z}(R)]_{\times}A) \\ &= \frac{[\mathcal{Z}(R) \times A\mathcal{Z}(R)]_{\times} + [\bar{A}\mathcal{Z}(\tilde{R})]_{\times}}{1 + \|\mathcal{Z}(R)\|^2}, \end{aligned}$$

where equalities $yx^{\top} - xy^{\top} = [x \times y]_{\times}$ and $M^{\top}[x]_{\times} + [x]_{\times}M + [Mx]_{\times} = \text{tr}(M)[x]_{\times}$, for all $x, y \in \mathbb{R}^3$ and $M \in \mathbb{R}^{3 \times 3}$, have been used. Consequently, one obtains

$$\begin{aligned} \psi(AR) &= \frac{\mathcal{Z}(R) \times A\mathcal{Z}(R) + \bar{A}\mathcal{Z}(\tilde{R})}{1 + \|\mathcal{Z}(R)\|^2} \\ &= \frac{(I - [\mathcal{Z}(R)]_{\times})}{1 + \|\mathcal{Z}(R)\|^2}\bar{A}\mathcal{Z}(R). \end{aligned}$$

It follows that

$$\begin{aligned} \|\psi(AR)\|^2 &= \frac{\mathcal{Z}(R)^{\top}\bar{A}(I + [\mathcal{Z}(R)]_{\times})(I - [\mathcal{Z}(R)]_{\times})\bar{A}\mathcal{Z}(R)}{(1 + \|\mathcal{Z}(R)\|^2)^2} \\ &= \frac{\mathcal{Z}(R)^{\top}\bar{A}(I - [\mathcal{Z}(R)]_{\times}^2)\bar{A}\mathcal{Z}(R)}{(1 + \|\mathcal{Z}(R)\|^2)^2} \\ &= \frac{\mathcal{Z}(R)^{\top}\bar{A}^2\mathcal{Z}(R)}{1 + \|\mathcal{Z}(R)\|^2} - \frac{(\mathcal{Z}(R)^{\top}\bar{A}\mathcal{Z}(R))^2}{(1 + \|\mathcal{Z}(R)\|^2)^2} \\ &= \frac{\|\bar{A}\mathcal{Z}(R)\|^2}{1 + \|\mathcal{Z}(R)\|^2} \left(1 - \frac{\|\mathcal{Z}(R)\|^2 \cos^2(\phi)}{1 + \|\mathcal{Z}(R)\|^2}\right), \end{aligned}$$

where $\phi = \angle(\mathcal{Z}(R), \bar{A}\mathcal{Z}(R))$. On the other hand, one has

$$\begin{aligned} \lambda_{\min}^{\bar{A}}\|\mathcal{Z}(R)\|^2 &\leq \mathcal{Z}(R)^{\top}\bar{A}\mathcal{Z}(R) = \|\mathcal{Z}(R)\|\|\bar{A}\mathcal{Z}(R)\|\cos(\phi) \\ &\leq \|\mathcal{Z}(R)\|^2\|\bar{A}\|_2\cos(\phi) = \lambda_{\max}^{\bar{A}}\|\mathcal{Z}(R)\|^2\cos(\phi), \end{aligned}$$

which implies that

$$\xi = \frac{\lambda_{\min}^{\bar{A}}}{\lambda_{\max}^{\bar{A}}} \leq \cos(\phi) \leq 1.$$

Consequently, it follows that

$$\begin{aligned} \|\psi(AR)\| &\leq (\lambda_{\max}^{\bar{A}})^2 \frac{\|\mathcal{Z}(R)\|^2}{1 + \|\mathcal{Z}(R)\|^2} \left(1 - \frac{\|\mathcal{Z}(R)\|^2 \xi^2}{1 + \|\mathcal{Z}(R)\|^2}\right) \\ &\geq (\lambda_{\min}^{\bar{A}})^2 \frac{\|\mathcal{Z}(R)\|^2}{1 + \|\mathcal{Z}(R)\|^2} \left(1 - \frac{\|\mathcal{Z}(R)\|^2}{1 + \|\mathcal{Z}(R)\|^2}\right), \end{aligned}$$

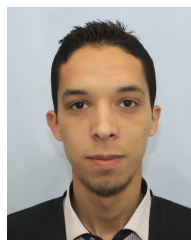
which implies identity (9) in view of the fact that

$$\|\mathcal{Z}(R)\|^2 = \frac{\|\psi(R)\|^2}{4(1 - |R|_I^2)^2} = \frac{|R|_I^2}{1 - |R|_I^2}. \quad (59)$$

REFERENCES

- [1] M. D. Shuster and S. D. Oh, "Three-axis attitude determination from vector observations," *Journal of Guidance and Control*, vol. 4, pp. 70–77, 1981.
- [2] F. Markley, "Attitude determination using vector observations and the singular value decomposition," *Journal of the Astronautical Sciences*, vol. 36, pp. 245–258, 1988.

- [3] —, “Attitude error representations for kalman filtering,” *Journal of Guidance, Control, and Dynamics*, vol. 63, no. 2, pp. 311–317, 2003.
- [4] J. Crassidis, F. Markley, and Y. Cheng, “Survey of nonlinear attitude estimation methods,” *Journal of Guidance Control and Dynamics*, vol. 30, no. 1, p. 12, 2007.
- [5] J. L. Crassidis and F. L. Markley, “Unscented filtering for spacecraft attitude estimation,” *Journal of guidance, control, and dynamics*, vol. 26, no. 4, pp. 536–542, 2003.
- [6] Carmi and Y. Oshman, “Adaptive particle filtering for spacecraft attitude estimation from vector observations,” *Journal of Guidance Control and Dynamics*, vol. 32, no. 1, pp. 232–241, 2009.
- [7] P. Corke, “An inertial and visual sensing system for a small autonomous helicopter,” *Journal of Robotic Systems*, vol. 21, no. 2, pp. 43–51, 2004.
- [8] A. Tayebi and S. McGilvray, “Attitude stabilization of a vtol quadrotor aircraft,” *IEEE Transactions on Control Systems Technology*, vol. 14, no. 3, pp. 562–571, 2006.
- [9] S. Salcudean, “A globally convergent angular velocity observer for rigid body motion,” *IEEE Transactions on Automatic Control*, vol. 36, no. 12, pp. 1493–1497, 1991.
- [10] B. Vik and T. I. Fossen, “A nonlinear observer for gps and ins integration,” in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 3. IEEE, 2001, pp. 2956–2961.
- [11] J. Thienel and R. Sanner, “A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 2011–2015, 2003.
- [12] S. Bonnabel, P. Martin, and P. Rouchon, “A non-linear symmetry-preserving observer for velocity-aided inertial navigation,” in *American Control Conference*, 2006, pp. 2910–2914.
- [13] A. Tayebi, S. McGilvray, A. Roberts, and M. Moallem, “Attitude estimation and stabilization of a rigid body using low-cost sensors,” in *Decision and Control, 2007 46th IEEE Conference on*. IEEE, 2007, pp. 6424–6429.
- [14] R. Mahony, T. Hamel, and J.-M. Pfimlin, “Nonlinear complementary filters on the special orthogonal group,” *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, June 2008.
- [15] J. F. Vasconcelos, C. Silvestre, and P. Oliveira, “A nonlinear GPS/IMU based observer for rigid body attitude and position estimation,” in *the 47th IEEE Conference on Decision and Control (CDC)*, 2008, pp. 1255–1260.
- [16] H. F. Grip, T. Fossen, T. A. Johansen, and A. Saberi, “Attitude estimation using biased gyro and vector measurements with time-varying reference vectors,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1332–1338, 2012.
- [17] A. Khosravian and M. Namvar, “Rigid body attitude control using a single vector measurement and gyro,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1273–1279, 2012.
- [18] M. Zamani, J. Trumpf, and R. Mahony, “Minimum-energy filtering for attitude estimation,” *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2917–2921, 2013.
- [19] M. Izadi and A. K. Sanyal, “Rigid body attitude estimation based on the lagrange-d’alembert principle,” *Automatica*, vol. 50, no. 10, pp. 2570–2577, 2014.
- [20] D. E. Zlotnik and J. R. Forbes, “Nonlinear estimator design on the special orthogonal group using vector measurements directly,” *IEEE Transactions on Automatic Control*, vol. 62, pp. 149–160, 2017.
- [21] D. E. Koditschek, “Application of a new lyapunov function to global adaptive attitude tracking,” in *The 27th Conference on Decision and Control, Austin, Texas*, 1988.
- [22] D. S. B. Sanjay P. Bhat, “A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon,” *Systems & Control Letters*, vol. 39, pp. 63–70, 2000.
- [23] P. Batista, C. Silvestre, and P. Oliveira, “Sensor-based globally asymptotically stable filters for attitude estimation: Analysis, design, and performance evaluation,” *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2095–2100, 2012.
- [24] —, “Globally exponentially stable cascade observers for attitude estimation,” *Control Engineering Practice*, vol. 20, no. 2, pp. 148–155, 2012.
- [25] C. G. Mayhew and A. R. Teel, “Hybrid control of rigid-body attitude with synergistic potential functions,” in *American Control Conference*, 2011, pp. 287–292.
- [26] T. Lee, “Global exponential attitude tracking controls on $SO(3)$,” *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2837–2842, 2015.
- [27] S. Berkane and A. Tayebi, “On the design of synergistic potential functions on $SO(3)$,” in *the 54th IEEE Conference on Decision and Control, Osaka, Japan*, 2015, pp. 270–275.
- [28] —, “Construction of synergistic potential functions on $SO(3)$ with application to velocity-free hybrid attitude stabilization,” *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 495–501, 2017.
- [29] E. K. Tse-Huai Wu and T. Lee, “Globally asymptotically stable attitude observer on $SO(3)$,” in *the 54th IEEE Conference on Decision and Control, Osaka, Japan*, 2015, pp. 2164–2168.
- [30] S. Berkane, A. Abdessameud, and A. Tayebi, “Global hybrid attitude estimation on the special orthogonal group $SO(3)$,” in *The 2016 American Control Conference, Boston, MA, USA*, pp. 113–118.
- [31] —, “A globally exponentially stable hybrid attitude and gyro-bias observer,” in *the 55th IEEE Conference on Decision and Control, Las Vegas, USA*, 2016, pp. 308–313.
- [32] —, “On the design of globally exponentially stable hybrid attitude and gyro-bias observers,” *Internal Technical Report, [Available online] arXiv:1605.05640v2*, 2016.
- [33] M. Izadi, E. Samiei, A. K. Sanyal, and V. Kumar, “Comparison of an attitude estimator based on the lagrange-d’alembert principle with some state-of-the-art filters,” in *2015 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2015, pp. 2848–2853.
- [34] D. E. Zlotnik and J. R. Forbes, “Exponential convergence of a nonlinear attitude estimator,” *Automatica*, vol. 72, pp. 11–18, 2016.
- [35] T. Lee, “Exponential stability of an attitude tracking control system on $SO(3)$ for large-angle rotational maneuvers,” *Systems & Control Letters*, vol. 61, no. 1, pp. 231–237, 2012.
- [36] A. Cayley, “Sur quelques propriétés des déterminants gauches,” *Journal für die reine und angewandte Mathematik*, vol. 32, pp. 119–123, 1846.
- [37] M. Shuster, “A survey of attitude representations,” *The Journal of the Astronautical Sciences*, vol. 41, no. 4, pp. 439–517, 1993.
- [38] E. D. Sontag and Y. Wang, “New Characterizations of Input-to-State Stability,” *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.
- [39] H. D. Black, “A passive system for determining the attitude of a satellite,” *AIAA Journal*, vol. 2, no. 7, pp. 1350–1351, 1964.
- [40] R. Mahony, T. Hamel, and J.-M. Pfimlin, “Complementary filter design on the special orthogonal group $SO(3)$,” in *The 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2005, pp. 1477–1484.
- [41] C. Lageman, J. Trumpf, and R. Mahony, “Gradient-like observers for invariant dynamics on a lie group,” *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 367–377, 2010.
- [42] A. Sanyal, A. Fosbury, N. Chaturvedi, and D. Bernstein, “Inertia-free spacecraft attitude tracking with disturbance rejection and almost global stabilization,” *Journal of Guidance, Control, Dynamics*, vol. 32, no. 4, pp. 1167–1178, 2009.
- [43] A. Tayebi, A. Roberts, and A. Benallegue, “Inertial measurements based dynamic attitude estimation and velocity-free attitude stabilization,” in *Proceedings of the 2011 American Control Conference*, June 2011, pp. 1027–1032.
- [44] A. Saccon, J. Hauser, and A. Aguiar, “Exploration of kinematic optimal control on the lie group $so(3)$,” in *8th IFAC Symposium on Nonlinear Control Systems*, 2010.
- [45] M. Krstić and Z.-H. Li, “Inverse optimal design of input-to-state stabilizing nonlinear controllers,” *IEEE Transactions on Automatic Control*, vol. 43, no. 3, pp. 336–350, 1998.
- [46] M.-d. Hua, G. Ducard, T. Hamel, R. Mahony, and K. Rudin, “Implementation of a Nonlinear Attitude Estimator for Aerial Robotic Vehicles,” *IEEE Transactions on Control Systems Technology*, vol. 22, no. 1, pp. 201–213, 2014.



Soulimane Berkane received his B.Sc. and M. Sc. degrees in Automatic Control from Ecole Nationale Polytechnique, Algiers, in 2013. He is currently a Ph. D. candidate and a Research Assistant at the department of Electrical and Computer Engineering at the University of Western Ontario, Canada. His research interest focuses on nonlinear and hybrid control with application to geometric attitude control and estimation.



Abdelhamid Tayebi received his B. Sc. in Electrical Engineering from Ecole Nationale Polytechnique, Algiers, in 1992, his M. Sc. in robotics from Université Pierre & Marie Curie, Paris, France in 1993, and his Ph. D. in Robotics and Automatic Control from Université de Picardie Jules Verne, France in December 1997. He joined the department of Electrical Engineering at Lakehead University in December 1999 where he is presently a full Professor. He is a Senior Member of IEEE and serves as an Associate Editor for *Automatica*, IEEE

Transactions on Control Systems Technology, *Control Engineering Practice* and IEEE CSS Conference Editorial Board. He also served as an Associate Editor for IEEE *Transactions on Cybernetics* (2006-2014). He is a member of the board of Directors of IFAC Canada. He is the founder and Director of the Automatic Control Laboratory at Lakehead University. His research interests are related to Control Engineering in general with applications to unmanned aerial vehicles.